

# Number and Algebra

IB HL Study Guide

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# IB Math AA HL — Number & Algebra

## Complete Study Guide

### Topics Covered

1. Sequences & Series — arithmetic and geometric, sigma notation, infinite sums
2. Exponents & Logarithms — laws, change of base, natural log, solving equations
3. Binomial Theorem — expansion, general term, finding specific terms
4. Proof by Induction **HL** — sum formulas, divisibility, formal structure
5. Systems of Equations **HL** — 3×3 systems, Gaussian elimination, geometric interpretation
6. Practice Questions & Exam Alerts

Topic 1 of the IB Math AA HL syllabus — Paper 1 and Paper 2

### IB TIP

**Formula booklet awareness:** The arithmetic and geometric sequence and series formulas, the binomial theorem, and the general binomial term formula are all in the IB formula booklet. However, the booklet does **not** provide the proof by induction structure or criteria for infinite geometric series convergence. Know those cold.

### MEMORISE THIS

#### Core formulas at a glance

Formula	Expression
Arithmetic $n$ th term	$u_n = u_1 + (n - 1)d$
Arithmetic sum	$S_n = \frac{n}{2}(2u_1 + (n - 1)d) = \frac{n}{2}(u_1 + u_n)$
Geometric $n$ th term	$u_n = u_1 \cdot r^{n-1}$
Geometric sum	$S_n = \frac{u_1(1 - r^n)}{1 - r}, r \neq 1$
Infinite geometric sum	$S_\infty = \frac{u_1}{1 - r},  r  < 1$
Binomial general term	$T_{r+1} = \binom{n}{r} a^{n-r} b^r$

## Section 1: Sequences & Series

A **sequence** is an ordered list of numbers following a rule. A **series** is the sum of the terms of a sequence. On the IB exam, sequences and series questions appear on both Paper 1 (no calculator) and Paper 2, and they frequently combine with logarithms or proofs in multi-part questions.

## 1.1 Arithmetic Sequences & Series

In an **arithmetic sequence**, each term is obtained by adding a fixed number  $d$  (the **common difference**) to the previous term.

$$u_n = u_1 + (n - 1)d$$

The sum of the first  $n$  terms of an arithmetic series:

$$S_n = \frac{n}{2}(2u_1 + (n - 1)d) \quad \text{or equivalently} \quad S_n = \frac{n}{2}(u_1 + u_n)$$

The second form is often faster when you already know the first and last terms.

### WORKED EXAMPLE

#### Arithmetic sequence — finding a term and sum

*The fourth term of an arithmetic sequence is 13 and the tenth term is 37. Find the common difference, the first term, and  $S_{20}$ .*

**Step 1:** Set up simultaneous equations using  $u_n = u_1 + (n - 1)d$ :

$$u_4 = u_1 + 3d = 13 \quad u_{10} = u_1 + 9d = 37$$

**Step 2:** Subtract the first equation from the second:

$$6d = 24 \implies d = 4$$

**Step 3:** Substitute back to find  $u_1$ :

$$u_1 + 3(4) = 13 \implies u_1 = 1$$

**Step 4:** Compute  $S_{20}$ :

$$S_{20} = \frac{20}{2}(2(1) + 19(4)) = 10(2 + 76) = 10 \times 78 = 780$$

### EXAM ALERT

**Common trap — arithmetic series:** Students often substitute  $n - 1$  instead of  $n$  when identifying the number of terms in a sum. For example, the sum from  $u_3$  to  $u_{10}$  contains  $10 - 3 + 1 = 8$  terms, not 7. Always count terms explicitly before using  $S_n$ .

## 1.2 Geometric Sequences & Series

In a **geometric sequence**, each term is obtained by multiplying the previous term by a fixed number  $r$  (the **common ratio**):

$$u_n = u_1 \cdot r^{n-1}$$

Sum of the first  $n$  terms (for  $r \neq 1$ ):

$$S_n = \frac{u_1(1-r^n)}{1-r}$$

An equivalent form sometimes used when  $r > 1$ :

$$S_n = \frac{u_1(r^n - 1)}{r - 1}$$

**Infinite geometric series:** When  $|r| < 1$ , the partial sums converge:

$$S_\infty = \frac{u_1}{1 - r}, \quad |r| < 1$$

### EXAM ALERT

**Critical condition:** You **must** state  $|r| < 1$  whenever you use  $S_\infty$ . An answer that uses the infinite sum formula without verifying or stating the convergence condition will lose marks in any IB mark scheme. If  $|r| \geq 1$ , the infinite sum does not exist.

### WORKED EXAMPLE

#### Geometric series — infinite sum and finding $r$

A geometric series has first term 12 and sum to infinity 20. Find the common ratio and the sum of the first 5 terms.

**Step 1:** Use  $S_\infty = \frac{u_1}{1 - r}$ :

$$20 = \frac{12}{1 - r} \implies 1 - r = \frac{12}{20} = 0.6 \implies r = 0.4$$

**Step 2:** Verify convergence:  $|0.4| < 1$  ✓

**Step 3:** Find  $S_5$ :

$$S_5 = \frac{12(1 - 0.4^5)}{1 - 0.4} = \frac{12(1 - 0.1024)}{0.6} = \frac{12 \times 0.8976}{0.6} = \frac{11.87712}{0.6} \approx 19.80$$

## 1.3 Sigma Notation

Sigma notation provides a compact way to write series:

$$\sum_{k=1}^n u_k = u_1 + u_2 + \cdots + u_n$$

Key properties:

$$\sum_{k=1}^n (a_k + b_k) = \sum_{k=1}^n a_k + \sum_{k=1}^n b_k \quad \sum_{k=1}^n c \cdot a_k = c \sum_{k=1}^n a_k$$

$$\sum_{k=1}^n c = nc \quad \sum_{k=1}^n k = \frac{n(n+1)}{2}$$

### IB TIP

**Reading sigma notation on Paper 1:** Always identify whether the index starts at 0 or 1 — it changes the number of terms. Also check whether the expression inside the sigma is arithmetic (linear in  $k$ ) or geometric (exponential in  $k$ ) to select the right formula.

## Section 2: Exponents & Logarithms

### 2.1 Laws of Exponents

For  $a > 0$ ,  $b > 0$ , and  $m, n \in \mathbb{R}$ :

Law	Expression
Product	$a^m \cdot a^n = a^{m+n}$
Quotient	$\frac{a^m}{a^n} = a^{m-n}$
Power of power	$(a^m)^n = a^{mn}$
Negative exponent	$a^{-n} = \frac{1}{a^n}$
Fractional exponent	$a^{m/n} = \sqrt[n]{a^m}$

### 2.2 Laws of Logarithms

For  $a > 0$ ,  $a \neq 1$ ,  $x > 0$ ,  $y > 0$ :

$$\log_a(xy) = \log_a x + \log_a y$$

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

$$\log_a(x^n) = n \log_a x$$

$$\log_a a = 1 \quad \log_a 1 = 0$$

**Change of base:**

$$\log_a x = \frac{\log_b x}{\log_b a} \quad (\text{commonly: } \log_a x = \frac{\ln x}{\ln a})$$

The **natural logarithm**  $\ln x = \log_e x$  where  $e \approx 2.718$ . On IB exams,  $\ln$  and  $e^x$  are inverse functions:  $e^{\ln x} = x$  and  $\ln(e^x) = x$ .

 **WORKED EXAMPLE**

**Solving an exponential equation**

Solve  $3^{2x+1} = 5^x$ , giving your answer in exact form.

**Step 1:** Take the natural log of both sides:

$$\ln(3^{2x+1}) = \ln(5^x)$$

**Step 2:** Apply the power law:

$$(2x + 1) \ln 3 = x \ln 5$$

**Step 3:** Expand and collect  $x$  terms:

$$2x \ln 3 + \ln 3 = x \ln 5 \quad \ln 3 = x \ln 5 - 2x \ln 3 = x(\ln 5 - 2 \ln 3)$$

**Step 4:** Solve for  $x$ :

$$x = \frac{\ln 3}{\ln 5 - 2 \ln 3} = \frac{\ln 3}{\ln 5 - \ln 9} = \frac{\ln 3}{\ln(5/9)}$$

 **EXAM ALERT**

**Log of a negative number:**  $\ln x$  and  $\log_a x$  are only defined for  $x > 0$ . After solving a logarithmic equation, always check that any proposed solution does not require taking the log of a non-positive number. Reject such extraneous solutions and state why.

 **MEMORISE THIS**

**Inverse relationships to memorise:**

$$a^{\log_a x} = x \quad \log_a(a^x) = x$$

$$e^{\ln x} = x \quad \ln(e^x) = x$$

These appear regularly in simplification and equation-solving steps.

## Section 3: Binomial Theorem

### 3.1 The Expansion

The **binomial theorem** states that for  $n \in \mathbb{Z}^+$ :

$$(a + b)^n = \sum_{r=0}^n \binom{n}{r} a^{n-r} b^r$$

where the **binomial coefficient** is:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Pascal's triangle provides  $\binom{n}{r}$  values for small  $n$ , but for larger  $n$  you should use the formula directly or the GDC.

### 3.2 The General Term

The  $(r + 1)$ th term (also written  $T_{r+1}$ ) of the expansion of  $(a + b)^n$  is:

$$T_{r+1} = \binom{n}{r} a^{n-r} b^r$$

Note the indexing:  $r$  starts at 0, so the **first term** is  $T_1$  (with  $r = 0$ ) and the **last term** is  $T_{n+1}$  (with  $r = n$ ).

#### IB TIP

**The most common binomial exam question:** “Find the term containing  $x^k$ ” or “find the coefficient of  $x^k$ .” The method is always: write the general term  $T_{r+1}$ , set the power of  $x$  equal to  $k$ , solve for  $r$ , then substitute back. Never expand the whole binomial if only one term is needed.

#### WORKED EXAMPLE

##### Finding a specific term in a binomial expansion

Find the term containing  $x^3$  in the expansion of  $\left(2x - \frac{1}{x}\right)^9$ .

**Step 1:** Write the general term with  $a = 2x$  and  $b = -\frac{1}{x}$ :

$$T_{r+1} = \binom{9}{r} (2x)^{9-r} \left(-\frac{1}{x}\right)^r = \binom{9}{r} 2^{9-r} (-1)^r \cdot x^{9-r} \cdot x^{-r}$$

**Step 2:** Combine powers of  $x$ :

$$T_{r+1} = \binom{9}{r} 2^{9-r} (-1)^r \cdot x^{9-2r}$$

**Step 3:** Set the power of  $x$  equal to 3:

$$9 - 2r = 3 \implies r = 3$$

**Step 4:** Substitute  $r = 3$ :

$$T_4 = \binom{9}{3} 2^6 (-1)^3 \cdot x^3 = 84 \times 64 \times (-1) \cdot x^3 = -5376 x^3$$

**Answer:** The term containing  $x^3$  is  $-5376x^3$ .

 **WORKED EXAMPLE**

**Binomial expansion with constant term**

Find the constant term in the expansion of  $\left(x^2 + \frac{3}{x}\right)^6$ .

**Step 1:** General term with  $a = x^2$  and  $b = \frac{3}{x}$ :

$$T_{r+1} = \binom{6}{r} (x^2)^{6-r} \left(\frac{3}{x}\right)^r = \binom{6}{r} 3^r \cdot x^{12-2r} \cdot x^{-r} = \binom{6}{r} 3^r \cdot x^{12-3r}$$

**Step 2:** For a constant term, set power of  $x$  to zero:

$$12 - 3r = 0 \implies r = 4$$

**Step 3:** Substitute  $r = 4$ :

$$T_5 = \binom{6}{4} 3^4 = 15 \times 81 = 1215$$

**Answer:** The constant term is 1215.

 **EXAM ALERT**

**Sign errors in binomial expansions:** When  $b$  is negative (e.g.,  $b = -\frac{1}{x}$ ), the sign alternates with  $r$ . Write  $(-1)^r$  explicitly in the general term; do not try to track signs mentally. One missed negative sign will cost the entire answer mark.

## Section 4: Proof by Induction HL

Proof by mathematical induction is an **HL-only** technique and is consistently tested on Paper 1. It proves that a statement  $P(n)$  holds for all integers  $n \geq n_0$  (usually  $n_0 = 1$ ).

### 4.1 The Formal Structure

Every induction proof must contain all four components:

1. **Base case:** Show  $P(n_0)$  is true by direct calculation.
2. **Inductive hypothesis:** State “Assume  $P(k)$  is true for some  $k \geq n_0$ ”, then write out explicitly what that assumption says.
3. **Inductive step:** Using the hypothesis, prove that  $P(k + 1)$  is true.
4. **Conclusion:** State “Therefore, by the principle of mathematical induction,  $P(n)$  is true for all  $n \geq n_0$ .”

 **EXAM ALERT**

**The most penalised induction error:** Students write “Assume true for  $n = k$ ” without saying what “true” means — i.e., without writing the actual formula for  $P(k)$ . IB mark schemes require the assumption to be written explicitly. If you omit the explicit statement of  $P(k)$ , you will lose the mark for the inductive hypothesis even if your algebra is correct.

 **MEMORISE THIS**

**Induction checklist for every proof:**

1. State  $P(n)$  (the claim) clearly at the top.
2. Base case: compute LHS and RHS separately, confirm they are equal (or the divisibility holds).
3. Write: “Assume  $P(k)$  is true, i.e., [write the formula/statement for  $k$ ].”
4. Show  $P(k + 1)$  follows. In the working, highlight exactly where the inductive hypothesis is substituted.
5. Write the conclusion in full — do not abbreviate it.

## 4.2 Worked Example: Sum of a Series

### WORKED EXAMPLE

Prove by induction that  $\sum_{r=1}^n r = \frac{n(n+1)}{2}$  for all  $n \in \mathbb{Z}^+$ .

State  $P(n)$ :  $\sum_{r=1}^n r = \frac{n(n+1)}{2}$

**Base case** ( $n = 1$ ):

$$\text{LHS} = 1. \text{ RHS} = \frac{1 \times 2}{2} = 1.$$

LHS = RHS, so  $P(1)$  is true. ✓

**Inductive hypothesis:** Assume  $P(k)$  is true for some  $k \in \mathbb{Z}^+$ , i.e.,

$$\sum_{r=1}^k r = \frac{k(k+1)}{2}$$

**Inductive step:** We need to show  $P(k+1)$  is true, i.e., that:

$$\sum_{r=1}^{k+1} r = \frac{(k+1)(k+2)}{2}$$

Starting from the LHS of  $P(k+1)$ :

$$\sum_{r=1}^{k+1} r = \underbrace{\sum_{r=1}^k r}_{\text{use hypothesis}} + (k+1) = \frac{k(k+1)}{2} + (k+1)$$

Factor out  $(k+1)$ :

$$= (k+1)\left(\frac{k}{2} + 1\right) = (k+1) \cdot \frac{k+2}{2} = \frac{(k+1)(k+2)}{2}$$

This is the RHS of  $P(k+1)$ . ✓

**Conclusion:** Since  $P(1)$  is true, and  $P(k) \Rightarrow P(k+1)$  for all  $k \in \mathbb{Z}^+$ , by the principle of mathematical induction  $P(n)$  is true for all  $n \in \mathbb{Z}^+$ . ■

### 4.3 Worked Example: Divisibility Proof

#### WORKED EXAMPLE

**Prove by induction that  $4^n - 1$  is divisible by 3 for all  $n \in \mathbb{Z}^+$ .**

**State  $P(n)$ :**  $3 \mid (4^n - 1)$ , i.e.,  $4^n - 1 = 3m$  for some integer  $m$ .

**Base case ( $n = 1$ ):**

$4^1 - 1 = 3 = 3 \times 1$ . This is divisible by 3.  $P(1)$  is true. ✓

**Inductive hypothesis:** Assume  $P(k)$  is true for some  $k \in \mathbb{Z}^+$ , i.e.,

$4^k - 1 = 3m$  for some integer  $m$ , so  $4^k = 3m + 1$

**Inductive step:** We need to show  $P(k + 1)$  is true, i.e.,  $3 \mid (4^{k+1} - 1)$ .

$4^{k+1} - 1 = 4 \cdot 4^k - 1$

Substitute the inductive hypothesis ( $4^k = 3m + 1$ ):

$= 4(3m + 1) - 1 = 12m + 4 - 1 = 12m + 3 = 3(4m + 1)$

Since  $4m + 1$  is an integer,  $4^{k+1} - 1$  is divisible by 3.  $P(k + 1)$  is true. ✓

**Conclusion:** Since  $P(1)$  is true, and  $P(k) \Rightarrow P(k + 1)$  for all  $k \in \mathbb{Z}^+$ , by the principle of mathematical induction  $4^n - 1$  is divisible by 3 for all  $n \in \mathbb{Z}^+$ . ■

#### IB TIP

**Divisibility proof strategy:** Isolate  $4^k$  (or the base raised to the  $k$ ) so you can substitute the inductive hypothesis directly. Then factorise so that the required divisor appears explicitly as a factor. Never skip the factoring step — writing " $\dots = 3(\dots)$ " is the payoff line that earns the mark.

## Section 5: Systems of Equations HL

### 5.1 Setting Up a 3×3 System

A system of three linear equations in three unknowns  $x, y, z$  can be written in **augmented matrix** form:

$$\left( \begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array} \right)$$

### 5.2 Gaussian Elimination

The goal is to reduce the augmented matrix to **row echelon form** (upper triangular) using three **elementary row operations** (EROs):

Operation	Notation	Description
Swap rows	$R_i \leftrightarrow R_j$	Interchange two rows
Scale a row	$kR_i \rightarrow R_i$	Multiply a row by a non-zero constant
Add multiple	$R_i + kR_j \rightarrow R_i$	Add a multiple of one row to another

Once in row echelon form, solve by **back substitution** from the bottom row up.

### WORKED EXAMPLE

#### Gaussian elimination — inconsistent system (no solution)

Solve:

$$x + 2y - z = 4 \quad 2x - y + 3z = 1 \quad 3x + y + 2z = 11$$

**Step 1:** Write the augmented matrix:

$$\left( \begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 2 & -1 & 3 & 1 \\ 3 & 1 & 2 & 11 \end{array} \right)$$

**Step 2:** Eliminate  $x$  from rows 2 and 3.

$$R_2 \rightarrow R_2 - 2R_1:$$

$$\left( \begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & -5 & 5 & -7 \\ 3 & 1 & 2 & 11 \end{array} \right)$$

$$R_3 \rightarrow R_3 - 3R_1:$$

$$\left( \begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & -5 & 5 & -7 \\ 0 & -5 & 5 & -1 \end{array} \right)$$

**Step 3:** Eliminate  $y$  from row 3.

$$R_3 \rightarrow R_3 - R_2:$$

$$\left( \begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & -5 & 5 & -7 \\ 0 & 0 & 0 & 6 \end{array} \right)$$

**Step 4:** The last row reads  $0 = 6$ , a **contradiction**. This system has **no solution** (the planes are inconsistent).

### IB TIP

**Adjusting the constants for unique vs. no vs. infinite solutions:** In exam questions, the system often has a parameter (e.g.,  $k$ ) in one entry. The three cases arise from the bottom row of the reduced matrix: if it gives  $0 = c$  with  $c \neq 0$ , no solution; if  $0 = 0$ , infinitely many solutions; if the diagonal is non-zero throughout, a unique solution exists.

## 5.3 Geometric Interpretation

Each equation  $ax + by + cz = d$  represents a **plane** in three-dimensional space. A  $3 \times 3$  system describes the intersection of three planes:

Reduced form outcome	Geometric meaning	Number of solutions
Full diagonal (no zeros)	Three planes meet at a single point	Unique solution
Row of zeros, consistent	Planes share a line or a plane	Infinitely many
Row $0\ 0\ 0 \mid c, c \neq 0$	Planes are inconsistent	No solution

### MEMORISE THIS

#### Recognising solution type from the augmented matrix:

- Last row all zeros  $\rightarrow \infty$  solutions (introduce a parameter  $t$ )
- Last row  $0\ 0\ 0 \mid c$  with  $c \neq 0 \rightarrow$  no solution
- All pivots non-zero  $\rightarrow$  unique solution

Always state which case applies — IB mark schemes award a mark for the correct interpretation.

## Section 6: Practice Questions

### Question 1 — Arithmetic series (Paper 1 style)

The sum of the first  $n$  terms of an arithmetic series is  $S_n = 3n^2 + 2n$ .

(a) Find the first term and the common difference.

(b) Find the 15th term.

► Show full solution

### Question 2 — Infinite geometric series

A geometric series has  $u_1 = 8$  and  $u_3 = 2$ .

(a) Find the two possible values of the common ratio.

(b) For the value of  $r$  for which the series converges, find  $S_\infty$ .

(c) Find the smallest value of  $n$  such that  $S_n > 0.99 S_\infty$ .

► Show full solution

### Question 3 — Binomial theorem

(a) Expand  $(1 + 2x)^5$  in ascending powers of  $x$  up to and including the term in  $x^3$ .

(b) Find the coefficient of  $x^4$  in the expansion of  $(3 - x)^7$ .

(c) **HL** Find the value of  $k$  such that the expansion of  $(k + x)^6$  has a term in  $x^2$  equal to  $60x^2$ .

► Show full solution

#### Question 4 — Proof by induction **HL**

Prove by mathematical induction that  $\sum_{r=1}^n r^2 = \frac{n(n+1)(2n+1)}{6}$  for all  $n \in \mathbb{Z}^+$ .

► Show full solution

#### Question 5 — Systems of equations **HL**

Consider the system of equations:

$$\begin{aligned} x + 2y + z &= 3 \\ 2x + y + az &= b \\ x - y + 2z &= 1 \end{aligned}$$

(a) Show that when  $a = 3$  and  $b = 4$ , the system has infinitely many solutions.

(b) Find the condition on  $a$  and  $b$  under which the system has a unique solution.

► Show full solution

#### Question 6 — Mixed: logarithms and series

The  $n$ th term of a sequence is  $u_n = \log_2(2^n \cdot 3)$ . Show that the sequence is arithmetic and find its common difference.

► Show full solution

#### EXAM ALERT

**PDF download:** A print-ready version of this guide with all formulas and worked examples is available at the link below.

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