IB Math AA HL Topic 3: Geometry and Trigonometry

IB HL Study Guide

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IB Math AA HL — Geometry & Trigonometry

Complete Study Guide

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Topic 3 of the IB Math AA HL syllabus — 25 SL hours, 51 HL hours — Paper 1 and Paper 2

IB TIP

Why this topic matters: Topic 3 has the most HL teaching hours (51) of any topic in Math AA HL. Vectors alone regularly appear as a 15–20 mark extended-response question on Paper 2, and trig identities appear almost every year on Paper 1 (no calculator). Mastering this topic is essential for a 6 or 7.

MEMORISE THIS

What is and is not in the formula booklet: The sine rule, cosine rule, area of a triangle formula, arc length and sector area formulas, the Pythagorean identity, double angle formulas, and compound angle formulas are all given. The unit circle exact values are NOT given — you must know $\sin$, $\cos$, and $\tan$ of $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ from memory. The cross product formula is given. Vector line and plane equations follow from definitions you must know.

Section 1: 3D Geometry

1.1 Distance and Midpoint in 3D

In three-dimensional space, points are described by coordinates (x, y, z). The distance between points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is a direct extension of the 2D Pythagorean theorem:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The **midpoint** M of segment AB is:

$$M = \left(\frac{x_1+x_2}{2},\ \frac{y_1+y_2}{2},\ \frac{z_1+z_2}{2}\right)$$

WORKED EXAMPLE

Distance in 3D

Find the distance between $P(1, -2, 4)$ and $Q(5, 0, 1)$, and the midpoint of PQ.

$$d = \sqrt{(5-1)^2 + (0-(-2))^2 + (1-4)^2} = \sqrt{16+4+9} = \sqrt{29}$$

$$M = \left(\frac{1+5}{2},\ \frac{-2+0}{2},\ \frac{4+1}{2}\right) = \left(3,\ -1,\ \frac{5}{2}\right)$$

1.2 Volumes and Surface Areas of 3D Solids

The IB expects you to apply standard formulae to composite 3D problems. All the following formulas are in the formula booklet.

MEMORISE THIS

Key 3D Solid Formulas

Solid	Volume	Surface Area
Cuboid ($l \times w \times h$)	lwh	$2(lw + lh + wh)$
Cylinder (radius r, height h)	$\pi r^2 h$	$2\pi r^2 + 2\pi rh$
Cone (radius r, height h, slant l)	$\frac{1}{3}\pi r^2 h$	$\pi r^2 + \pi rl$
Sphere (radius r)	$\frac{4}{3}\pi r^3$	$4\pi r^2$
Pyramid (base area A, height h)	$\frac{1}{3}Ah$	Base + lateral faces

Note: slant height of cone $l = \sqrt{r^2 + h^2}$.

⚠ **EXAM ALERT**

Composite solids: Many IB questions join two solids (e.g., a cone on top of a cylinder). For volume, add the parts. For surface area, subtract any internal faces that are no longer external. A cone sitting on a cylinder shares a circular face — the base circle of the cone and the top circle of the cylinder cancel each other from the surface area count.

WORKED EXAMPLE

Composite Solid — Cone on Cylinder

A solid consists of a cylinder of radius 4 cm and height 6 cm, with a cone of the same base radius and height 3 cm placed on top. Find the total volume and the total surface area.

Volume:

$V = \pi(4)^2(6) + \frac{1}{3}\pi(4)^2(3) = 96\pi + 16\pi = 112\pi \approx 352\ \text{cm}^3$

Surface area: The solid has a circular base (bottom of cylinder), the curved lateral surface of the cylinder, and the curved lateral surface of the cone. The circle where cylinder meets cone is internal — it does not contribute.

Slant height of cone: $l = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$ cm

$SA = \pi(4)^2 + 2\pi(4)(6) + \pi(4)(5) = 16\pi + 48\pi + 20\pi = 84\pi \approx 264\ \text{cm}^2$

Section 2: Right-Angle and Non-Right-Angle Trigonometry

2.1 Right-Triangle Trigonometry (SOH CAH TOA)

For a right-angled triangle with acute angle θ, hypotenuse h, opposite side o, and adjacent side a:

$\sin\theta = \frac{o}{h} \qquad \cos\theta = \frac{a}{h} \qquad \tan\theta = \frac{o}{a}$

These definitions apply directly to right-triangle problems (ladders, building heights, simple bearings).

2.2 The Sine Rule

For any triangle with sides a, b, c opposite to angles A, B, C respectively:

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Use the sine rule when you know:

- Two angles and one side (AAS or ASA), or
- Two sides and a non-included angle (SSA — watch for the **ambiguous case**)

⚠ EXAM ALERT

The ambiguous case (SSA): When you are given two sides and a non-included angle, the equation $\frac{\sin B}{b} = \frac{\sin A}{a}$ can sometimes give two valid solutions for angle B (one acute, one obtuse — supplementary angles have the same sine). Always check whether $B' = 180^\circ - B$ also produces a valid triangle before concluding there is only one solution.

WORKED EXAMPLE

Sine Rule — Finding a Side

In triangle ABC, angle $A = 35^\circ$, angle $B = 72^\circ$, and side $a = 8.4$ cm. Find side b.

First find $C = 180^\circ - 35^\circ - 72^\circ = 73^\circ$.

$\frac{b}{\sin B} = \frac{a}{\sin A}$

$b = \frac{8.4 \times \sin 72^\circ}{\sin 35^\circ} = \frac{8.4 \times 0.9511}{0.5736} \approx 13.9 \text{ cm}$

2.3 The Cosine Rule

For a triangle with sides a, b, c and angle C opposite side c:

$c^2 = a^2 + b^2 - 2ab \cos C$

Rearranged to find angle C when all three sides are known:

$\cos C = \frac{a^2+b^2-c^2}{2ab}$

Use the cosine rule when you know:

- Two sides and the **included** angle (SAS), or
- All three sides (SSS)

WORKED EXAMPLE

Cosine Rule — Finding an Angle

A triangle has sides $a = 7, b = 5, c = 9$. Find angle C.

$\cos C = \frac{7^2+5^2-9^2}{2(7)(5)} = \frac{49+25-81}{70} = \frac{-7}{70} = -0.1$

$C = \arccos(-0.1) \approx 95.7^\circ$

Since $\cos C < 0$, the angle C is obtuse — this is consistent with c being the longest side.

2.4 Area of a Triangle

$\text{Area} = \frac{1}{2}ab\sin C$

where a and b are two sides and C is the included angle between them.

IB TIP

When to use which formula: If you have base and height, use $\frac{1}{2}bh$. If you have two sides and the included angle, use $\frac{1}{2}ab\sin C$. If you have all three sides, use Heron's formula (though this is less common on IB exams). The formula $\frac{1}{2}ab\sin C$ is the most useful in non-right-angle contexts.

Section 3: Applications — Bearings and Angles of Elevation/Depression

3.1 Bearings

A **bearing** is a direction measured as a three-digit angle clockwise from North. For example, a bearing of 065° means 65° clockwise from North (i.e., roughly north-east).

Key conventions:

- Bearings are always measured **clockwise** from **North**
- They are written as three digits: $005^\circ, 090^\circ, 180^\circ, 270^\circ$
- A bearing of 090° is due East; 180° is due South; 270° is due West

Reverse bearing: If the bearing from A to B is θ, the bearing from B to A is $\theta + 180^\circ \pmod{360^\circ}$.

WORKED EXAMPLE

Bearings Problem

A ship travels 12 km on a bearing of 050°*, then 9 km on a bearing of* 140°*. Find the distance and bearing back to the start.*

The two legs form a triangle. The angle between the directions 050° and 140° is $140^\circ - 50^\circ = 90^\circ$ (the turn at the intermediate point is 90°, making the paths perpendicular).

$d = \sqrt{12^2 + 9^2} = \sqrt{144 + 81} = \sqrt{225} = 15 \text{ km}$

The angle at the start satisfies $\tan\theta = \frac{9}{12} = 0.75$, so $\theta = 36.87^\circ \approx 36.9^\circ$.

Bearing from start to final position: $050^\circ + 36.9^\circ = 086.9^\circ \approx 087^\circ$.

Return bearing: $087^\circ + 180^\circ = 267^\circ$.

3.2 Angles of Elevation and Depression

- **Angle of elevation:** The angle measured upward from the horizontal to a line of sight.
- **Angle of depression:** The angle measured downward from the horizontal to a line of sight.

These angles are equal when the observer and the object are at the same horizontal level for alternate interior angles, but in general they are supplementary parts of the right-angle trig setup.

⚠ EXAM ALERT

Common setup: The angle of depression from the top of a cliff to a boat is equal to the angle of elevation from the boat to the top of the cliff (alternate angles with parallel horizontal lines). Labelling a clear diagram always prevents errors in these problems. Always draw and label a diagram before writing any equation.

Section 4: Radian Measure, Arc Length, and Sector Area

4.1 Radian Measure

One **radian** is the angle subtended at the centre of a circle by an arc equal in length to the radius.

$\pi \text{ radians} = 180^\circ$

Conversions:

$\theta\ (\text{radians}) = \theta\ (\text{degrees}) \times \frac{\pi}{180^\circ}$ $\theta\ (\text{degrees}) = \theta\ (\text{radians}) \times \frac{180^\circ}{\pi}$

MEMORISE THIS

Common angle conversions

Degrees	Radians
0°	0
30°	$\frac{\pi}{6}$
45°	$\frac{\pi}{4}$
60°	$\frac{\pi}{3}$
90°	$\frac{\pi}{2}$
120°	$\frac{2\pi}{3}$
135°	$\frac{3\pi}{4}$
150°	$\frac{5\pi}{6}$
180°	π
270°	$\frac{3\pi}{2}$
360°	2π

4.2 Arc Length and Sector Area

For a circle of radius r and a sector with central angle θ (in **radians**):

Arc length $\quad l = r\theta$

Sector area $\quad A = \frac{1}{2}r^2\theta$

Segment area = Sector area − Triangle area:

$A_{\text{segment}} = \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta = \frac{1}{2}r^2(\theta - \sin\theta)$

⚠ **EXAM ALERT**

These formulas only work when θ is in **radians**. If the angle is given in degrees, convert first. A common exam trap is to leave the angle in degrees and use the arc length formula directly — this will always give the wrong answer.

WORKED EXAMPLE

Arc Length and Sector Area

A sector of a circle has radius 8 cm and central angle $\frac{5\pi}{6}$ *radians. Find the arc length, the sector area, and the area of the corresponding minor segment.*

Arc length: $l = r\theta = 8 \times \frac{5\pi}{6} = \frac{40\pi}{6} = \frac{20\pi}{3} \approx 20.9 \text{ cm}$

Sector area: $A = \frac{1}{2}r^2\theta = \frac{1}{2}(64)\left(\frac{5\pi}{6}\right) = \frac{320\pi}{12} = \frac{80\pi}{3} \approx 83.8 \text{ cm}^2$

Segment area: $A_{\text{seg}} = \frac{1}{2}r^2(\theta - \sin\theta) = 32\left(\frac{5\pi}{6} - \sin\frac{5\pi}{6}\right) = 32\left(\frac{5\pi}{6} - \frac{1}{2}\right) \approx 32(2.618 - 0.5) \approx 67.8 \text{ cm}^2$

Section 5: The Unit Circle and Exact Trigonometric Values

5.1 The Unit Circle

The **unit circle** has centre $(0, 0)$ and radius 1. For any angle θ measured anticlockwise from the positive x-axis, the point on the unit circle is $(\cos\theta, \sin\theta)$.

This gives a definition of sin and cos that extends beyond right triangles to all real angles.

$\cos\theta = x\text{-coordinate}$ $\quad$ $\sin\theta = y\text{-coordinate}$ $\quad$ $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x}$

5.2 Signs in Each Quadrant (CAST)

Quadrant	Angles	Positive functions
1st (0 to $\frac{\pi}{2}$)	0° to 90°	All ($\sin, \cos, \tan$)
2nd ($\frac{\pi}{2}$ to π)	90° to 180°	sin only
3rd (π to $\frac{3\pi}{2}$)	180° to 270°	tan only
4th ($\frac{3\pi}{2}$ to 2π)	270° to 360°	cos only

The acronym **CAST** (reading quadrants 4–1–2–3 anticlockwise) is a mnemonic: Cos, All, Sin, Tan.

5.3 Exact Trigonometric Values

MEMORISE THIS

Exact values — must know from memory

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	undefined

Memory trick for sin**:** The values for sin at $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ are $\frac{\sqrt{0}}{2}, \frac{\sqrt{1}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{4}}{2}$ — a neat pattern under the square root.

5.4 Symmetry and Reference Angles

For any angle θ in quadrants $2, 3$, or 4, find the **reference angle** α (the acute angle to the nearest x-axis), then apply the sign from CAST.

- Quadrant 2: $\alpha = \pi - \theta$
- Quadrant 3: $\alpha = \theta - \pi$
- Quadrant 4: $\alpha = 2\pi - \theta$

Example: $\sin\left(\frac{5\pi}{6}\right) = \sin\left(\pi - \frac{\pi}{6}\right) = \sin\frac{\pi}{6} = \frac{1}{2}$ (positive in Q2).

Section 6: Trigonometric Identities

6.1 Pythagorean Identity

From the unit circle definition ($\cos^2\theta + \sin^2\theta = 1$ is immediate from $x^2 + y^2 = 1$):

$\sin^2\theta + \cos^2\theta = 1$

Two derived forms (divide through by $\cos^2\theta$ or $\sin^2\theta$):

$1 + \tan^2\theta = \sec^2\theta \qquad \cot^2\theta + 1 = \csc^2\theta$

6.2 Double Angle Identities

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

IB TIP

The three forms of $\cos 2\theta$ are all in the booklet, but knowing when to use each saves time. Use $2\cos^2\theta - 1$ when you want to eliminate sin; use $1 - 2\sin^2\theta$ when you want to eliminate cos; use $\cos^2\theta - \sin^2\theta$ as the starting point for proofs.

6.3 Compound Angle Identities HL

$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$

$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$

Note the sign convention: for cos, the signs are **opposite** to the signs in the angle.

WORKED EXAMPLE

Compound Angle Identity — Exact Value

Show that $\sin 75^\circ = \dfrac{\sqrt{6}+\sqrt{2}}{4}$.

Write $75^\circ = 45^\circ + 30^\circ$:

$\sin(45^\circ + 30^\circ) = \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$

$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$

$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{6}+\sqrt{2}}{4}$ ✓

WORKED EXAMPLE

Identity Proof

Prove that $\dfrac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$.

Starting from the left-hand side:

$\frac{\sin 2\theta}{1+\cos 2\theta} = \frac{2\sin\theta\cos\theta}{1+(2\cos^2\theta-1)} = \frac{2\sin\theta\cos\theta}{2\cos^2\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta$ ✓

EXAM ALERT

Identity proofs: Always start from one side (usually the more complex side) and work towards the other. Never manipulate both sides simultaneously — the IB mark scheme expects a one-directional chain of equalities. Write "LHS $= \cdots = \cdots =$ RHS" explicitly. Starting from the left and reaching the right shows the most logical flow.

Section 7: Trigonometric Functions and Their Graphs

7.1 Graphs of sin, cos, and tan

The three basic trig functions have the following key features:

Feature	$y = \sin x$	$y = \cos x$	$y = \tan x$
Period	2π	2π	π
Amplitude	1	1	undefined
Domain	$\mathbb{R}$	$\mathbb{R}$	$x \neq \frac{\pi}{2} + n\pi, n \in \mathbb{Z}$
Range	$[-1, 1]$	$[-1, 1]$	$\mathbb{R}$
y-intercept	0	1	0
Zeros	$n\pi$	$\frac{\pi}{2} + n\pi$	$n\pi$

7.2 Transformations of Trig Functions

The general form is $y = a\sin(b(x-c)) + d$ (and similarly for cos and tan):

- $|a|$ = **amplitude** (vertical stretch/compression; if $a < 0$, reflection in x-axis)
- $\frac{2\pi}{|b|}$ = **period** (for sin and cos); $\frac{\pi}{|b|}$ for tan
- c = **horizontal shift** (phase shift; positive c shifts right)
- d = **vertical shift** (midline is at $y = d$)

WORKED EXAMPLE

Graph Transformations

Describe the transformations of $y = -3\cos\left(2x - \frac{\pi}{3}\right) + 1$ *and state the amplitude, period, phase shift, and range.*

Rewrite as $y = -3\cos\left(2\left(x - \frac{\pi}{6}\right)\right) + 1$.

- $a = -3$: amplitude 3, reflected in x-axis
- $b = 2$: period $= \frac{2\pi}{2} = \pi$
- $c = \frac{\pi}{6}$: shifted right by $\frac{\pi}{6}$
- $d = 1$: shifted up by 1; midline $y = 1$

Range: $[1 - 3,\ 1 + 3] = [-2, 4]$

EXAM ALERT

Factoring b before reading the phase shift: You must write $b(x - c)$, not $bx - k$. To find the phase shift from $y = \cos(2x - \frac{\pi}{3})$, factor out the 2: $y = \cos\left(2\left(x - \frac{\pi}{6}\right)\right)$, so the phase shift is $\frac{\pi}{6}$, not $\frac{\pi}{3}$. This is one of the most common errors in trig graph questions.

Section 8: Solving Trigonometric Equations

8.1 General Strategy

To solve a trig equation on a given interval:

1. Isolate the trig function (e.g., $\sin x = \frac{1}{2}$)
2. Find the principal value using inverse trig
3. Use the unit circle or CAST diagram to find all solutions in the interval
4. Check that all solutions satisfy any restrictions

General solutions (all solutions, no interval restriction):

$$\sin x = k \implies x = \arcsin k + 2n\pi \quad \text{or} \quad x = \pi - \arcsin k + 2n\pi, \quad n \in \mathbb{Z}$$

$$\cos x = k \implies x = \pm\arccos k + 2n\pi, \quad n \in \mathbb{Z}$$

$$\tan x = k \implies x = \arctan k + n\pi, \quad n \in \mathbb{Z}$$

WORKED EXAMPLE

Solving a Trig Equation on an Interval

Solve $2\cos^2 x - \cos x - 1 = 0$ *for* $0 \le x \le 2\pi$.

Step 1: Factorise — treat $\cos x$ as the variable.

$2\cos^2 x - \cos x - 1 = (2\cos x + 1)(\cos x - 1) = 0$

Step 2: Set each factor to zero.

$\cos x = -\frac{1}{2}$ or $\cos x = 1$

Step 3: Solve each case on $[0, 2\pi]$.

For $\cos x = 1$: $x = 0$ (and $x = 2\pi$, but check interval is $0 \le x \le 2\pi$, so $x = 0$ and $x = 2\pi$ are both valid endpoints).

For $\cos x = -\frac{1}{2}$: reference angle $= \frac{\pi}{3}$; cosine is negative in Q2 and Q3.

$x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ or $x = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$

Solutions: $x \in \left\{0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi\right\}$

WORKED EXAMPLE

Trig Equation Requiring an Identity

Solve $\sin 2x = \sin x$ *for* $0 \le x < 2\pi$.

Step 1: Apply the double angle identity: $\sin 2x = 2\sin x \cos x$.

$2\sin x \cos x = \sin x$

Step 2: Rearrange (do not divide by $\sin x$ — you would lose solutions!).

$2\sin x \cos x - \sin x = 0$

$\sin x(2\cos x - 1) = 0$

Step 3: Solve each factor.

$\sin x = 0 \implies x = 0, \ \pi$

$\cos x = \frac{1}{2} \implies x = \frac{\pi}{3}, \frac{5\pi}{3}$

Solutions: $x \in \left\{0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}\right\}$

⚠ **EXAM ALERT**

Never divide both sides of a trig equation by a trig function (e.g., dividing $\sin x(2\cos x - 1) = 0$ by $\sin x$). Dividing loses the solutions where $\sin x = 0$. Always factorise and set each factor to zero instead.

Section 9: Reciprocal and Inverse Trigonometric Functions HL

9.1 Reciprocal Trig Functions

$\csc\theta = \frac{1}{\sin\theta}$ $\sec\theta = \frac{1}{\cos\theta}$ $\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$

These are defined wherever the denominator is non-zero. Key identities:

$1 + \tan^2\theta = \sec^2\theta$ $\cot^2\theta + 1 = \csc^2\theta$

MEMORISE THIS

Graphs of reciprocal functions: Each has vertical asymptotes wherever the original function equals zero.

- $y = \csc x$: asymptotes at $x = n\pi$; range $(-\infty, -1] \cup [1, \infty)$
- $y = \sec x$: asymptotes at $x = \frac{\pi}{2} + n\pi$; range $(-\infty, -1] \cup [1, \infty)$
- $y = \cot x$: asymptotes at $x = n\pi$; period π; range $\mathbb{R}$

9.2 Inverse Trigonometric Functions HL

To make trig functions invertible, their domains are restricted:

Function	Restricted domain	Range (principal values)
$\arcsin x = \sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\arccos x = \cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\arctan x = \tan^{-1} x$	$\mathbb{R}$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

⚠ **EXAM ALERT**

Inverse trig outputs principal values only. Your calculator gives one answer. When solving equations, use the principal value to find all solutions with the unit circle — do not assume the calculator output is the only answer. For example, $\arcsin(0.5) = \frac{\pi}{6}$, but the equation $\sin x = 0.5$ has infinitely many solutions.

WORKED EXAMPLE

Reciprocal Trig Identity Proof

Prove that $\dfrac{\csc\theta}{\cot\theta + \tan\theta} = \cos\theta$.

LHS:

$$\frac{\csc\theta}{\cot\theta+\tan\theta} = \frac{\frac{1}{\sin\theta}}{\frac{\cos\theta}{\sin\theta}+\frac{\sin\theta}{\cos\theta}}$$

Simplify the denominator using a common denominator of $\sin\theta\cos\theta$:

$$= \frac{\frac{1}{\sin\theta}}{\frac{\cos^2\theta+\sin^2\theta}{\sin\theta\cos\theta}} = \frac{\frac{1}{\sin\theta}}{\frac{1}{\sin\theta\cos\theta}} = \frac{1}{\sin\theta} \times \sin\theta\cos\theta = \cos\theta \qquad \checkmark$$

Section 10: Vector Algebra HL

10.1 Vector Notation and Basic Operations

A **vector** has both magnitude and direction. In 2D and 3D, vectors are written as column vectors or using unit vector notation:

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}$$

where $\mathbf{i} = (1,0,0), \mathbf{j} = (0,1,0), \mathbf{k} = (0,0,1)$ are standard unit vectors.

Magnitude (modulus):

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

Unit vector in the direction of $\mathbf{v}$:

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

Vector between two points: $\overrightarrow{AB} = B - A = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix}$

10.2 Vector Addition and Scalar Multiplication

$$\mathbf{u}+\mathbf{v} = \begin{pmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{pmatrix} \qquad k\mathbf{v} = \begin{pmatrix} kv_1 \\ kv_2 \\ kv_3 \end{pmatrix}$$

Geometrically, vector addition follows the triangle or parallelogram law. Scalar multiplication stretches or compresses the vector (and flips it if $k < 0$).

Parallel vectors: $\mathbf{u}$ and $\mathbf{v}$ are parallel if $\mathbf{u} = k\mathbf{v}$ for some scalar k.

10.3 The Dot Product (Scalar Product)

$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 = |\mathbf{u}||\mathbf{v}|\cos\theta$

where θ is the angle between the vectors. Rearranging gives:

$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$

Key property: $\mathbf{u} \cdot \mathbf{v} = 0$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are **perpendicular** (provided both are non-zero).

WORKED EXAMPLE

Dot Product — Angle Between Vectors

Find the angle between $\mathbf{u} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ *and* $\mathbf{v} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$.

$\mathbf{u} \cdot \mathbf{v} = (2)(1) + (-1)(4) + (3)(-2) = 2 - 4 - 6 = -8$

$|\mathbf{u}| = \sqrt{4+1+9} = \sqrt{14} \qquad |\mathbf{v}| = \sqrt{1+16+4} = \sqrt{21}$

$\cos\theta = \frac{-8}{\sqrt{14}\cdot\sqrt{21}} = \frac{-8}{\sqrt{294}}$

$\theta = \arccos\left(\frac{-8}{\sqrt{294}}\right) \approx 117.8^\circ$

10.4 The Cross Product (Vector Product) HL

The cross product $\mathbf{u} \times \mathbf{v}$ produces a vector **perpendicular** to both $\mathbf{u}$ and $\mathbf{v}$.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix}$$

Magnitude of the cross product:

$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta$

This equals the **area of the parallelogram** formed by $\mathbf{u}$ and $\mathbf{v}$. The area of the triangle formed by $\mathbf{u}$ and $\mathbf{v}$ is $\dfrac{1}{2}|\mathbf{u} \times \mathbf{v}|$.

Key property: $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are **parallel**.

EXAM ALERT

Cross product is not commutative: $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$. The direction of the result follows the right-hand rule. On IB exams, the question will often ask for a normal vector to a plane — the cross product of two vectors in the plane gives exactly that.

WORKED EXAMPLE

Cross Product — Normal Vector and Area

Given $A(1,0,2)$, $B(3,1,0)$, $C(0,2,1)$, find a vector normal to the plane ABC and the area of triangle ABC.

$$\overrightarrow{AB} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \qquad \overrightarrow{AC} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} (1)(-1) - (-2)(2) \\ (-2)(-1) - (2)(-1) \\ (2)(2) - (1)(-1) \end{pmatrix} = \begin{pmatrix} -1+4 \\ 2+2 \\ 4+1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$$

Normal vector: $\mathbf{n} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$

Area of triangle: $A = \frac{1}{2}\left|\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}\right| = \frac{1}{2}\sqrt{9+16+25} = \frac{\sqrt{50}}{2} = \frac{5\sqrt{2}}{2}$

Section 11: Vector Equations of Lines HL

11.1 Vector Equation of a Line

A line in 2D or 3D is determined by a **point** on the line and a **direction vector**. If the line passes through point A with position vector $\mathbf{a}$ and has direction vector $\mathbf{d}$, then any point P on the line satisfies:

$$\mathbf{r} = \mathbf{a} + t\mathbf{d}, \qquad t \in \mathbb{R}$$

where t is a scalar parameter. In component form (3D):

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + t\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

11.2 Parametric and Cartesian Forms

Parametric equations:

$$x = a_1 + td_1 \qquad y = a_2 + td_2 \qquad z = a_3 + td_3$$

Cartesian (symmetric) form (provided $d_i \neq 0$):

$$\frac{x-a_1}{d_1} = \frac{y-a_2}{d_2} = \frac{z-a_3}{d_3} = t$$

11.3 Angle Between Two Lines

The angle θ between lines with direction vectors $\mathbf{d}_1$ and $\mathbf{d}_2$:

$\cos\theta = \frac{|\mathbf{d}_1 \cdot \mathbf{d}_2|}{|\mathbf{d}_1||\mathbf{d}_2|}$

The absolute value ensures $\theta \in [0^\circ, 90^\circ]$ (we take the acute angle between lines).

11.4 Relationships Between Lines

Two lines can be:

- **Parallel:** direction vectors are scalar multiples; lines do not intersect (unless they are the same line)
- **Intersecting:** lines meet at a unique point (solve simultaneously for parameters)
- **Skew:** lines are not parallel and do not intersect (only possible in 3D)

To check if two lines intersect: set the parametric equations equal and solve for the parameters s and t. If all three equations are consistent with specific values of s and t, the lines intersect. If the system is inconsistent, the lines are skew (or parallel).

WORKED EXAMPLE

Line Intersection in 3D

Line L_1*:* $\mathbf{r} = \begin{pmatrix}1\\2\\0\end{pmatrix} + s\begin{pmatrix}1\\-1\\2\end{pmatrix}$ *and line* L_2*:* $\mathbf{r} = \begin{pmatrix}3\\0\\4\end{pmatrix} + t\begin{pmatrix}-1\\1\\0\end{pmatrix}$*. Determine whether the lines intersect, are parallel, or are skew.*

Step 1: Check if direction vectors are parallel. $(1, -1, 2)$ and $(-1, 1, 0)$ are not scalar multiples (since $\frac{1}{-1} = -1$ but $\frac{2}{0}$ is undefined), so the lines are not parallel.

Step 2: Set corresponding components equal.

$1 + s = 3 - t \quad \Rightarrow \quad s + t = 2 \qquad (i)$ $2 - s = t \quad \Rightarrow \quad s + t = 2 \qquad (ii)$ $0 + 2s = 4 \quad \Rightarrow \quad s = 2 \qquad (iii)$

From (iii): $s = 2$. From (i): $t = 0$.

Step 3: Verify in the z-component of L_2: $z = 4 + 0(0) = 4$. For L_1: $z = 0 + 2(2) = 4$. Consistent.

The lines intersect at the point: L_1 at $s = 2$: $(1 + 2, 2 - 2, 0 + 4) = (3, 0, 4)$. Intersection point: $(3, 0, 4)$.

⚠ EXAM ALERT

Skew lines: After finding s and t from two components, always verify in the third component. If the third equation is inconsistent, the lines are skew — not parallel and not intersecting. This check is essential and frequently omitted by students, leading to incorrect conclusions.

Section 12: Vector Equations of Planes HL

12.1 Equation of a Plane

A **plane** is determined by a point and a **normal vector** perpendicular to it.

If $\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is the normal vector and the plane passes through point (x_0, y_0, z_0), the **Cartesian equation** is:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Expanded as $ax + by + cz = d$ where $d = ax_0 + by_0 + cz_0$.

Vector form: $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$, or equivalently $\mathbf{n} \cdot \mathbf{r} = d$.

Parametric form: Using two non-parallel vectors $\mathbf{b}$ and $\mathbf{c}$ in the plane:

$$\mathbf{r} = \mathbf{a} + s\mathbf{b} + t\mathbf{c}, \qquad s, t \in \mathbb{R}$$

12.2 Finding the Equation of a Plane

Given a normal vector and a point: Substitute directly into $ax + by + cz = d$.

Given three points: Find two vectors in the plane, take their cross product to get the normal, then use one point to find d.

✎ WORKED EXAMPLE

Equation of a Plane Through Three Points

Find the Cartesian equation of the plane through $P(1,0,2)$, $Q(3,1,0)$, $R(0,2,1)$.

Step 1: Find two vectors in the plane.

$$\overrightarrow{PQ} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \qquad \overrightarrow{PR} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$$

Step 2: Cross product gives the normal.

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{pmatrix} (1)(-1) - (-2)(2) \\ (-2)(-1) - (2)(-1) \\ (2)(2) - (1)(-1) \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$$

Step 3: Use point $P(1,0,2)$ to find d.

$d = 3(1) + 4(0) + 5(2) = 3 + 0 + 10 = 13$

Equation of the plane: $3x + 4y + 5z = 13$

Verification with $Q(3,1,0)$*:* $9 + 4 + 0 = 13$ ✓. *With* $R(0,2,1)$*:* $0 + 8 + 5 = 13$ ✓.

12.3 Angle Between a Line and a Plane

Let $\mathbf{d}$ be the direction of the line and $\mathbf{n}$ be the normal to the plane. The angle ϕ between the line and the plane satisfies:

$\sin\phi = \frac{|\mathbf{d}\cdot\mathbf{n}|}{|\mathbf{d}||\mathbf{n}|}$

(Note: this is the complement of the angle between the line and the normal.)

12.4 Angle Between Two Planes

The angle between two planes is the angle between their normal vectors:

$\cos\theta = \frac{|\mathbf{n}_1\cdot\mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|}$

💡 IB TIP

Normal vector reading: From a Cartesian plane equation $ax + by + cz = d$, the normal vector is immediately $\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. You do not need to find two vectors in the plane and cross them — just read the coefficients. This saves significant time on exams.

Section 13: Intersections of Lines and Planes HL

13.1 Line and Plane Intersection

To find where a line $\mathbf{r} = \mathbf{a} + t\mathbf{d}$ meets a plane $ax + by + cz = k$:

1. Substitute the parametric equations of the line into the plane equation
2. Solve for t
3. Substitute t back to find the intersection point

If $\mathbf{d} \cdot \mathbf{n} = 0$, the line is parallel to the plane (no intersection unless the line lies in the plane).

WORKED EXAMPLE

Line-Plane Intersection

Find the point where $L\colon \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ *meets the plane* $\Pi\colon x + 2y - z = 5$

.

Parametric equations of L**:** $x = 1 + 2t,\ y = 2 - t,\ z = -1 + 3t$

Substitute into plane equation:

$(1 + 2t) + 2(2 - t) - (-1 + 3t) = 5$

$1 + 2t + 4 - 2t + 1 - 3t = 5$

$6 - 3t = 5 \implies t = \frac{1}{3}$

Intersection point: $x = 1 + \frac{2}{3} = \frac{5}{3}, y = 2 - \frac{1}{3} = \frac{5}{3}, z = -1 + 1 = 0.$

Point of intersection: $\left(\dfrac{5}{3}, \dfrac{5}{3}, 0\right)$

13.2 Intersection of Two Planes

Two non-parallel planes intersect in a **line**. The line's direction is given by $\mathbf{n}_1 \times \mathbf{n}_2$ (normal vectors of the two planes). To find a specific point on the line, set one variable to 0 and solve the two plane equations simultaneously.

13.3 Intersection of Three Planes

Three planes can intersect in:

- A unique point (most common exam case — solve the 3×3 system)
- A line (two planes have the same intersection line, and the third contains it)
- No point (inconsistent system — parallel planes or other configurations)

Use row reduction (Gaussian elimination) to solve the 3×3 system. The number of solutions corresponds to the rank of the augmented matrix.

WORKED EXAMPLE

Intersection of Three Planes

Find the intersection of the planes $\Pi_1 : x + y + z = 6$, $\Pi_2 : 2x - y + z = 3$, $\Pi_3 : x + 2y - z = 4$.

Set up the system:

$$\begin{aligned} x + y + z &= 6 \\ 2x - y + z &= 3 \\ x + 2y - z &= 4 \end{aligned}$$

Eliminate x: Using $R_2 \leftarrow R_2 - 2R_1$ and $R_3 \leftarrow R_3 - R_1$:

$$\begin{aligned} x + y + z &= 6 \\ -3y - z &= -9 \\ y - 2z &= -2 \end{aligned}$$

From row 2: $3y + z = 9$ so $z = 9 - 3y$.

Substitute into row 3: $y - 2(9 - 3y) = -2 \implies y - 18 + 6y = -2 \implies 7y = 16 \implies y = \frac{16}{7}$

Then: $z = 9 - 3\left(\frac{16}{7}\right) = 9 - \frac{48}{7} = \frac{15}{7}$, $x = 6 - y - z = 6 - \frac{16}{7} - \frac{15}{7} = 6 - \frac{31}{7} = \frac{11}{7}$

Intersection point: $\left(\frac{11}{7}, \frac{16}{7}, \frac{15}{7}\right)$

EXAM ALERT

Inconsistency check: When solving three plane equations, if elimination leads to a contradiction like $0 = 5$, the system is inconsistent — the planes do not all meet at a common point (they form a prism or triangular arrangement). If elimination produces $0 = 0$, the planes all contain a common line. Always interpret the geometric meaning of the outcome in your written answer.

Section 14: Practice Exam Questions

IB TIP

Exam strategy for Topic 3: In Papers 1 and 2, trig identities and solving trig equations appear on Paper 1 (no calculator). Use exact values. Vectors, 3D geometry, and applications appear on Paper 2. Always show full working, including labelled diagrams for geometry problems. Vectors questions often follow a chain: find a normal, write a plane equation, find an intersection — plan the whole question before starting.

WORKED EXAMPLE

Question 1 [Non-Right-Angle Trig and Sector] [~8 marks]

Triangle ABC has $AB = 10$ cm, $AC = 7$ cm, and angle $BAC = 1.2$ radians.

(a) Find the area of triangle ABC.

(b) Find the length BC.

(c) A sector of a circle has the same area as triangle ABC and a radius of 5 cm. Find the central angle of the sector in radians.

▶ Show Solution

WORKED EXAMPLE

Question 2 [Trig Identity Proof and Equation] [~9 marks]

(a) Prove that $\cos 2\theta + 2\sin^2\theta = 1$.

(b) Hence, or otherwise, solve $\cos 4x + 2\sin^2 2x - 1 = 0$ for $0 \leq x \leq \pi$.

▶ Show Solution

WORKED EXAMPLE

Question 3 [Compound Angle — HL] [~9 marks]

(a) Given that $\sin\alpha = \frac{3}{5}$ and $\cos\beta = \frac{5}{13}$, where α and β are both acute angles, find the exact value of $\cos(\alpha + \beta)$.

(b) Show that $\cos(\alpha + \beta) = \cos(\alpha - \beta) - 2\sin\alpha\sin\beta$.

▶ Show Solution

WORKED EXAMPLE

Question 4 [Vectors — Lines and Planes] [~15 marks] HL

The plane Π_1 passes through the points $A(2,0,1)$, $B(1,3,-1)$, and $C(0,1,2)$.

(a) Find the equation of plane Π_1 in the form $ax + by + cz = d$.

A second plane Π_2 has equation $2x - y + z = 4$.

(b) Find the angle between Π_1 and Π_2.

(c) Find the vector equation of the line L of intersection of Π_1 and Π_2.

A point P lies on L with $x = 0$.

(d) Find the coordinates of P.

▶ Show Solution

WORKED EXAMPLE

Question 5 [3D Trig Application] [~9 marks]

A vertical tower AB stands on horizontal ground at A. From a point C on the ground, the angle of elevation of the top B is 42°. The bearing of A from C is 325°, and $CA = 80$ m. A second point D on the ground is 50 m due East of C.

(a) Find the height AB of the tower.

(b) Find the angle of elevation of B from D.

(c) Find the bearing of A from D.

▶ Show Solution

MEMORISE THIS

Topic 3 Quick-Reference Summary

Concept	Key Formula / Fact
Distance in 3D	$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
Sine rule	$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
Cosine rule	$c^2 = a^2 + b^2 - 2ab\cos C$
Triangle area	$\frac{1}{2}ab\sin C$
Arc length	$l = r\theta$ (θ in radians)
Sector area	$A = \frac{1}{2}r^2\theta$
Segment area	$\frac{1}{2}r^2(\theta - \sin\theta)$
$\sin^2\theta + \cos^2\theta$	$= 1$
$\cos 2\theta$ (3 forms)	$\cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
$\sin 2\theta$	$2\sin\theta\cos\theta$
$\sin(A \pm B)$	$\sin A\cos B \pm \cos A\sin B$
$\cos(A \pm B)$	$\cos A\cos B \mp \sin A\sin B$
Dot product	$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 = \lvert\mathbf{u}\rvert\lvert\mathbf{v}\rvert\cos\theta$
Perpendicular vectors	$\mathbf{u} \cdot \mathbf{v} = 0$
Angle between vectors	$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\lvert\mathbf{u}\rvert\lvert\mathbf{v}\rvert}$
Plane from normal $\mathbf{n} = (a, b, c)$	$ax + by + cz = d$
Line equation	$\mathbf{r} = \mathbf{a} + t\mathbf{d}$
Normal to plane ABC	$\overrightarrow{AB} \times \overrightarrow{AC}$
Angle between planes	$\cos\theta = \frac{\lvert\mathbf{n}_1 \cdot \mathbf{n}_2\rvert}{\lvert\mathbf{n}_1\rvert\lvert\mathbf{n}_2\rvert}$