Functions

IB HL Study Guide

Contents

How to Use This Guide

Functions is Topic 2 of the IB Math AA HL syllabus and is one of the highest-yield topics for Paper 1 (non-calculator). The concepts here underpin all of calculus, statistics, and algebra — you cannot afford gaps. This guide covers subtopics 2.1–2.5 in syllabus order plus the AHL extension: language of functions, composite and inverse functions, transformations, rational functions, the modulus function, and polynomial division with the factor and remainder theorems. Every section includes worked examples drawn from past IB papers, exam alert boxes flagging the Paper 1 traps that cost marks most often, and MCQ practice at the end of each section. Use the jump links below to navigate directly to the section you need.

MEMORISE THIS

Formula booklet reminder: The booklet provides the general forms of transformations, but not the rules for determining domains of composites or restrictions for invertibility. You must know those cold. The booklet also does not give you asymptote rules — degree comparison and long division are your tools.

Section 1: Language of Functions (2.1)

A **function** is a rule that assigns to each element of a set (the **domain**) exactly one element of another set (the **codomain**). The set of all output values actually produced is called the **range** (or image). Note: the range is a subset of the codomain, but they are not necessarily equal.

$f : X \to Y$

where X is the domain, Y is the codomain, and the range is $\{f(x) : x \in X\} \subseteq Y$.

1.1 Key Vocabulary

Term	Definition	Notation
Domain	Set of permitted input values	$\text{dom}(f)$ or X
Codomain	Set within which outputs must fall	Y
Range / Image	Set of actual output values produced	$\text{range}(f)$
Image of a point	The specific output for a given input	$f(a)$
Pre-image	Input(s) that map to a given output	$f^{-1}(\{b\})$

Arrow diagram notation: In an arrow diagram, each arrow begins at a domain element and ends at the corresponding codomain element. A mapping is a function if and only if every domain element has exactly one outgoing arrow.

1.2 Types of Function

One-to-one (injective): Every element of the range is the image of exactly one domain element. No two different inputs produce the same output. Equivalently: $f(a) = f(b) \Rightarrow a = b$.

Many-to-one: At least one output value is the image of more than one input. Example: $f(x) = x^2$ maps both $x = 2$ and $x = -2$ to 4.

Onto (surjective): The range equals the codomain — every element of the codomain is the image of at least one domain element.

Bijective: Both injective and surjective. Only bijective functions have a well-defined inverse over their full domain.

IB TIP

Horizontal line test: A function is one-to-one if and only if every horizontal line intersects its graph at most once. This test is Paper 1-ready — you can apply it from a sketch in under five seconds.

⚠ EXAM ALERT

A common mark-losing error: confusing **range** and **codomain**. If $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2$, the codomain is $\mathbb{R}$, but the range is $[0, \infty)$. The IB mark scheme treats these as distinct — write "range" when you mean the set of actual outputs.

1.3 Finding the Domain and Range

Natural domain: the largest subset of $\mathbb{R}$ for which $f(x)$ is defined. Restrictions arise from:

- **Square roots:** the expression under the radical must be ≥ 0.
- **Denominators:** the denominator must be $\neq 0$.
- **Logarithms:** the argument must be > 0.

WORKED EXAMPLE

Finding domain and range

Find the domain and range of $f(x) = \sqrt{4 - x^2}$.

Domain: Require $4 - x^2 \geq 0$, i.e. $x^2 \leq 4$, so $-2 \leq x \leq 2$.

$\text{Domain: } [-2,\ 2]$

Range: The expression $\sqrt{4 - x^2}$ is the upper half of a circle of radius 2. The maximum occurs at $x = 0$ (giving $\sqrt{4} = 2$) and the minimum is 0 at $x = \pm 2$.

$\text{Range: } [0,\ 2]$

Section 2: Composite and Inverse Functions (2.2)

2.1 Composite Functions

The **composite function** $f \circ g$ (read "f composed with g") is defined by:

$$(f \circ g)(x) = f(g(x))$$

Think of g as the inner function (applied first) and f as the outer function (applied second). Order matters: $f \circ g \neq g \circ f$ in general.

Domain of $f \circ g$: The domain is the set of all x in $\text{dom}(g)$ such that $g(x) \in \text{dom}(f)$.

$$\text{dom}(f \circ g) = \{x \in \text{dom}(g) : g(x) \in \text{dom}(f)\}$$

WORKED EXAMPLE

Finding a composite and its domain

Let $f(x) = \sqrt{x}$ *(domain* $x \geq 0$*) and* $g(x) = 3 - x^2$ *(domain* $\mathbb{R}$*). Find* $(f \circ g)(x)$ *and its domain.*

Step 1: $(f \circ g)(x) = f(g(x)) = \sqrt{3 - x^2}$

Step 2: Domain requires $3 - x^2 \geq 0$, so $x^2 \leq 3$, giving $-\sqrt{3} \leq x \leq \sqrt{3}$.

$\text{dom}(f \circ g) = [-\sqrt{3},\ \sqrt{3}]$

EXAM ALERT

On Paper 1, the most common composite-function trap is applying the functions in the wrong order. The notation $(f \circ g)(x) = f(g(x))$ means g **acts first**. Write out both functions separately before composing — one mis-step and every subsequent mark is lost.

2.2 Inverse Functions

If f is **one-to-one (injective)**, then the **inverse function** f^{-1} is defined so that:

$$f^{-1}(f(x)) = x \quad \text{for all } x \in \text{dom}(f) \; f(f^{-1}(x)) = x \quad \text{for all } x \in \text{dom}(f^{-1})$$

Geometrically, the graph of f^{-1} is the reflection of the graph of f in the line $y = x$.

Key relationships:

- $\text{dom}(f^{-1}) = \text{range}(f)$
- $\text{range}(f^{-1}) = \text{dom}(f)$

Algorithm for finding $f^{-1}(x)$:

1. Write $y = f(x)$.
2. Rearrange to express x in terms of y.
3. Swap x and y (replace every y with x and every x with y).
4. The result is $y = f^{-1}(x)$.
5. State the domain of f^{-1} (which equals the range of f).

WORKED EXAMPLE

Finding an inverse function

Find $f^{-1}(x)$ *for* $f(x) = \dfrac{2x+1}{x-3}$, $x \neq 3$.

Step 1: Let $y = \dfrac{2x+1}{x-3}$.

Step 2: Rearrange for x:

$$y(x-3) = 2x+1 \; xy - 3y = 2x + 1 \; xy - 2x = 3y + 1 \; x(y-2) = 3y+1 \; x = \tfrac{3y+1}{y-2}$$

Step 3: Swap $x \leftrightarrow y$:

$$f^{-1}(x) = \tfrac{3x+1}{x-2}, \quad x \neq 2$$

Verification: $f(f^{-1}(x)) = f\left(\dfrac{3x+1}{x-2}\right) = \dfrac{2 \cdot \frac{3x+1}{x-2} + 1}{\frac{3x+1}{x-2} - 3} = \dfrac{\frac{6x+2+x-2}{x-2}}{\frac{3x+1-3x+6}{x-2}} = \dfrac{7x}{7} = x$ ✓

2.3 Restricting the Domain for Invertibility

A function that is not one-to-one does not have an inverse over its full domain. You can, however, **restrict the domain** to a maximal interval on which the function is one-to-one, then find the inverse on that restricted domain.

WORKED EXAMPLE

Restricting domain of $f(x) = x^2$

Find a restricted domain and the inverse of $f(x) = x^2$.

$f(x) = x^2$ is many-to-one over $\mathbb{R}$ (fails horizontal line test), so it has no inverse on $\mathbb{R}$.

Restriction: $\mathrm{dom}(f) = [0, \infty)$ makes f one-to-one.

Inverse: $y = x^2 \Rightarrow x = \sqrt{y}$ (taking the positive root because $x \geq 0$).

$$f^{-1}(x) = \sqrt{x}, \quad x \geq 0$$

IB TIP

When a question asks "state the largest domain for which f has an inverse," look for the largest interval containing the given point on which the function is monotone (strictly increasing or strictly decreasing). For a parabola with vertex at $x = a$, the two standard choices are $x \geq a$ and $x \leq a$.

Section 3: Transformations of Functions (2.3)

A transformation maps a parent function $y = f(x)$ to a new function by shifting, reflecting, or stretching its graph. The IB AA HL syllabus requires you to identify and apply six transformation types fluently.

3.1 Transformation Summary Table

Transformation	Function form	Effect on graph
Horizontal translation right by a	$f(x-a)$	Every point moves a units right
Horizontal translation left by a	$f(x+a)$	Every point moves a units left
Vertical translation up by b	$f(x)+b$	Every point moves b units up
Vertical translation down by b	$f(x)-b$	Every point moves b units down
Reflection in the y-axis	$f(-x)$	x-coordinates change sign
Reflection in the x-axis	$-f(x)$	y-coordinates change sign
Vertical stretch by factor a	$a\,f(x)$	y-coordinates multiplied by a
Horizontal stretch by factor $\frac{1}{b}$	$f(bx)$	x-coordinates divided by b

MEMORISE THIS

Inside vs outside the function:

- Changes **inside** the brackets (to x) affect horizontal behaviour — and they do the **opposite** of what you expect: $f(x-3)$ moves **right**, not left.
- Changes **outside** the brackets (to the whole expression) affect vertical behaviour — and they do exactly what you expect: $f(x)+3$ moves **up** by 3.

This asymmetry is the single most-tested transformation concept on IB Paper 1.

3.2 Combining Transformations

When multiple transformations are combined, apply them in the correct order:

1. Horizontal stretch/compression (changes to x inside brackets)
2. Horizontal translation
3. Vertical stretch/compression
4. Vertical translation

For $y = a\,f(b(x-h)) + k$:

- $|b| > 1$: horizontal compression (narrowing)
- $0 < |b| < 1$: horizontal stretch (widening)
- $|a| > 1$: vertical stretch (tallening)
- $0 < |a| < 1$: vertical compression (flattening)
- h: horizontal shift right by h (left if $h < 0$)
- k: vertical shift up by k (down if $k < 0$)

WORKED EXAMPLE

Describing a combined transformation

Describe the transformations that map $y = x^2$ *onto* $y = 3(x-2)^2 + 1$.

Reading from the form $y = a\,f(b(x-h)) + k$ with $f(x) = x^2$, $a = 3, b = 1, h = 2$, $k = 1$:

1. Vertical stretch by factor 3 (every y-value multiplied by 3).
2. Translation by $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ (2 units right, 1 unit up).

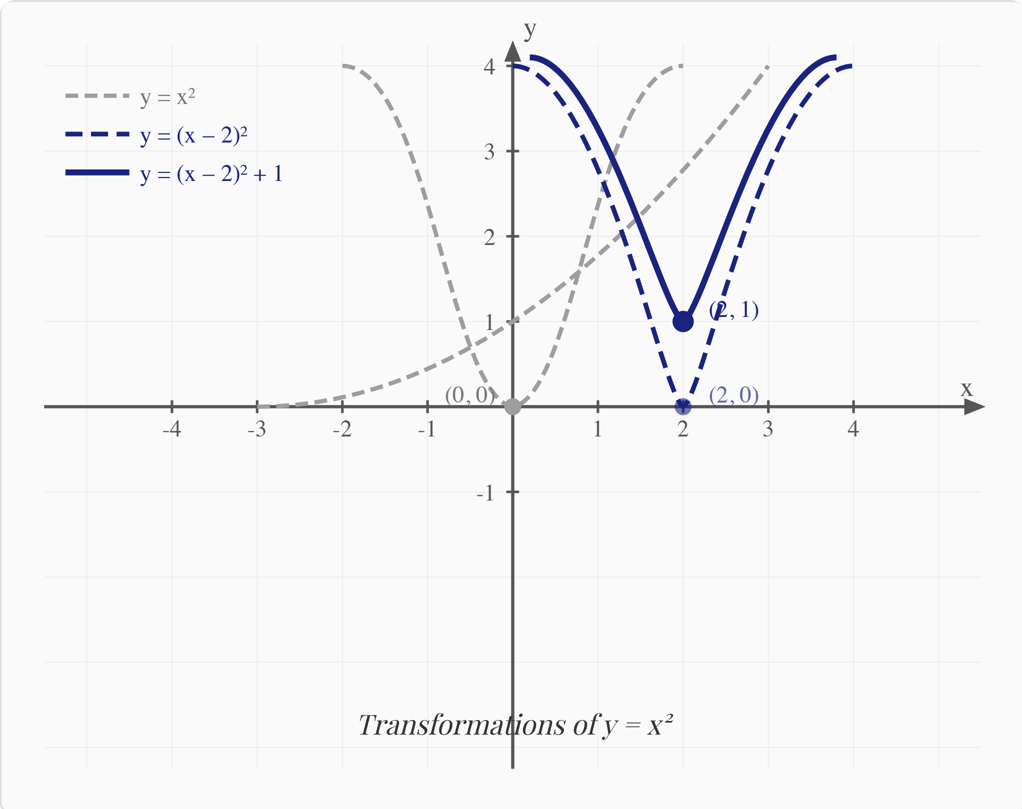
Note: vertical stretch before translation, or the translation remains correct regardless of order here since $b = 1$.

EXAM ALERT

Translation vectors are frequently asked on Paper 1. Write translation as a column vector: $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ means 2 right, 1 up. The IB mark scheme awards the mark only for the correct vector form — "shift right 2 and up 1" in words may not receive full credit if vector form is specifically requested.

3.3 Transformation Diagram

The diagram below shows how the transformations $y = x^2 \to y = (x-2)^2 \to y = (x-2)^2 + 1$ shift the parabola.



The parabola $y = x^2$ (grey dashed) is shifted 2 units right to give $y = (x-2)^2$ (navy dashed), then 1 unit up to give $y = (x-2)^2 + 1$ (solid navy). Vertex moves from $(0,0)$ to $(2,0)$ to $(2,1)$.

3.4 Invariant Points

A point is **invariant** under a transformation if it maps to itself. For reflections in the x-axis, invariant points satisfy $f(x) = 0$ (points on the x-axis). For reflections in the line $y = x$ (which corresponds to finding f^{-1}), invariant points satisfy $f(x) = x$.

⚠ **EXAM ALERT**

Finding invariant points under $y = f^{-1}(x)$ means solving $f(x) = x$, **not** $f^{-1}(x) = f(x)$. These are the same equation rearranged, but students often confuse them and solve the wrong equation.

Section 4: Rational Functions (2.4)

A **rational function** has the form $f(x) = \dfrac{p(x)}{q(x)}$ where p and q are polynomials and $q(x) \not\equiv 0$. The key features are asymptotes and holes.

4.1 Vertical Asymptotes

A **vertical asymptote** at $x = a$ occurs where the denominator is zero and the numerator is non-zero. The function grows without bound as $x \to a$.

Finding vertical asymptotes:

1. Factor the denominator fully.
2. Find zeros of the denominator.
3. Check whether the numerator is also zero at those points.
   - If numerator $\neq 0$: vertical asymptote at $x = a$.
   - If numerator $= 0$: there is a **hole** (removable discontinuity) at $x = a$, not an asymptote.

4.2 Horizontal Asymptotes

A **horizontal asymptote** at $y = L$ occurs when $f(x) \to L$ as $x \to \pm\infty$.

Let $\deg p = m$ and $\deg q = n$:

$$y = \begin{cases} 0 & m < n \\ \dfrac{\text{leading coeff of } p}{\text{leading coeff of } q} & m = n \\ \text{no horizontal asymptote} & m > n \end{cases}$$

When $m > n$, divide p by q using polynomial long division to find an **oblique (slant) asymptote** of the form $y = ax + b$.

MEMORISE THIS

Asymptote rules in one sentence: "Bottom zero gives vertical; top and bottom same degree gives a horizontal fraction; top one degree higher gives oblique via long division."

4.3 Oblique Asymptotes HL

When $\deg p = \deg q + 1$, divide to obtain:

$f(x) = ax + b + \frac{r(x)}{q(x)}$

where r is the remainder. As $x \to \pm\infty$, the remainder term $\to 0$, so $y = ax + b$ is the oblique asymptote.

WORKED EXAMPLE

Oblique asymptote by long division

Find the oblique asymptote of $f(x) = \dfrac{x^2 + 3x - 2}{x + 1}$.

Perform long division of $x^2 + 3x - 2$ by $x + 1$:

$x^2 + 3x - 2 = (x + 1)(x + 2) - 4$

Therefore:

$f(x) = x + 2 - \frac{4}{x+1}$

As $x \to \pm\infty$, $\dfrac{4}{x + 1} \to 0$, so the oblique asymptote is $y = x + 2$.

4.4 Sketching Rational Functions

Systematic approach for $f(x) = \dfrac{p(x)}{q(x)}$**:**

1. Find all **vertical asymptotes** (zeros of q where $p \neq 0$).
2. Find all **holes** (zeros of q where $p = 0$ too — cancel the common factor first).
3. Find the **horizontal or oblique asymptote** using degree comparison / long division.
4. Find x-intercepts: set $p(x) = 0$ (after cancellation).
5. Find y-intercept: compute $f(0)$.
6. Determine sign of f in each interval between vertical asymptotes using a sign table.
7. Sketch, ensuring the curve approaches each asymptote from the correct side.

WORKED EXAMPLE

Sketching $f(x) = \dfrac{x+1}{x-2}$

Step 1 — Vertical asymptote: $x - 2 = 0 \Rightarrow x = 2$. Numerator at $x = 2$: $3 \neq 0$. Vertical asymptote: $x = 2$.

Step 2 — Holes: No common factors. No holes.

Step 3 — Horizontal asymptote: Both numerator and denominator degree 1; leading coefficients both 1. Horizontal asymptote: $y = 1$.

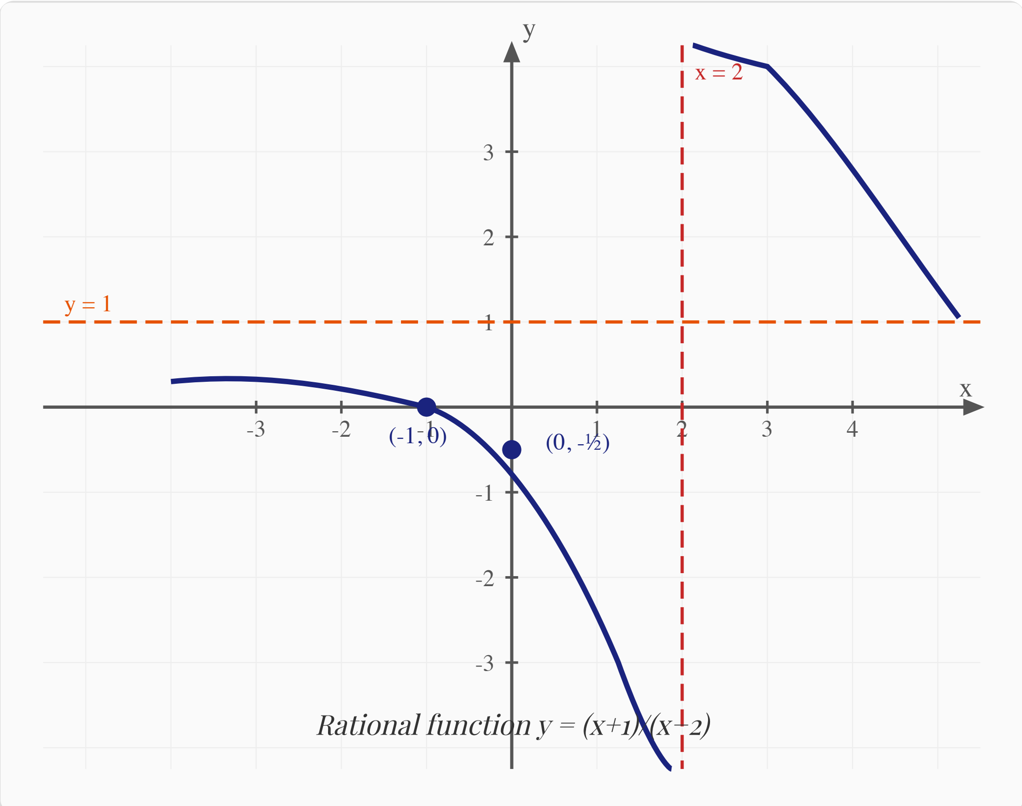
Step 4 — x-intercept: $x + 1 = 0 \Rightarrow x = -1$. Point: $(-1, 0)$.

Step 5 — y-intercept: $f(0) = \dfrac{1}{-2} = -\dfrac{1}{2}$. Point: $\left(0, -\dfrac{1}{2}\right)$.

Step 6 — Sign table:

Interval	Sign of $x+1$	Sign of $x-2$	Sign of f
$x < -1$	$-$	$-$	$+$
$-1 < x < 2$	$+$	$-$	$-$
$x > 2$	$+$	$+$	$+$

4.5 Rational Function Diagram



Graph of $y = \dfrac{x+1}{x-2}$. Vertical asymptote (red dashed) at $x = 2$; horizontal asymptote (orange dashed) at $y = 1$. The curve crosses the x-axis at $(-1, 0)$ and the y-axis at $(0, -\frac{1}{2})$.

⚠ **EXAM ALERT**

On Paper 1 sketching questions, you **must** label all asymptotes with their equations and mark all intercepts with coordinates. A sketch without labelled asymptotes will lose the asymptote marks even if the curve shape is correct. Write $x = a$ on vertical asymptotes and $y = b$ on horizontal asymptotes.

Section 5: Modulus Function and Inequalities (2.5)

The **modulus function** (absolute value) is defined as:

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

Geometrically, $|x|$ is the distance of x from the origin on the number line.

5.1 Solving Equations Involving Modulus

Basic form $|f(x)| = k$ (where $k > 0$):

$$|f(x)| = k \implies f(x) = k \quad \text{or} \quad f(x) = -k$$

Always **check both solutions** by substituting back into the original equation, especially when the original problem places restrictions on the domain.

WORKED EXAMPLE

Solving $|2x - 3| = 5$

Case 1: $2x - 3 = 5 \Rightarrow 2x = 8 \Rightarrow x = 4$

Case 2: $2x - 3 = -5 \Rightarrow 2x = -2 \Rightarrow x = -1$

Both solutions are valid. Answer: $x = 4$ or $x = -1$.

WORKED EXAMPLE

Solving $|x - 1| = |2x + 3|$

When both sides are moduli, square both sides (valid since both sides are ≥ 0):

$(x-1)^2 = (2x+3)^2$ $x^2 - 2x + 1 = 4x^2 + 12x + 9$ $3x^2 + 14x + 8 = 0$ $(3x + 2)(x + 4) = 0$ $x = -\frac{2}{3} \quad \text{or} \quad x = -4$

Verify: $|-\frac{2}{3} - 1| = \frac{5}{3}$ and $|2(-\frac{2}{3}) + 3| = \frac{5}{3}$ ✓. $|-4-1| = 5$ and $|2(-4)+3| = 5$ ✓.

5.2 Modulus Inequalities

Form $|f(x)| < k$ (strict; $k > 0$):

$$|f(x)| < k \iff -k < f(x) < k$$

Form $|f(x)| > k$ (strict; $k > 0$):

$$|f(x)| > k \iff f(x) > k \quad \text{or} \quad f(x) < -k$$

MEMORISE THIS

Inequality direction rule: $|f(x)| < k$ gives a **bounded** (finite) interval — it is an "and" condition. $|f(x)| > k$ gives an **unbounded** region — it is an "or" condition (two separate intervals). Confusing these two directions is the most common modulus error on Paper 1.

WORKED EXAMPLE

Solving $|3x - 6| \leq 9$

$-9 \leq 3x - 6 \leq 9 \; -3 \leq 3x \leq 15 \; -1 \leq x \leq 5$

Answer: $x \in [-1, 5]$.

WORKED EXAMPLE

Solving $|x + 2| > 4$

Branch 1: $x + 2 > 4 \Rightarrow x > 2$

Branch 2: $x + 2 < -4 \Rightarrow x < -6$

Answer: $x \in (-\infty, -6) \cup (2, \infty)$.

5.3 Graphs of $|f(x)|$ and $f(|x|)$

These two operations produce different graphs and are a common source of confusion.

$y = |f(x)|$**:** Take the graph of $y = f(x)$, then **reflect all parts below the x-axis upward**. Parts above the axis remain unchanged.

$y = f(|x|)$**:** Take the right half of $y = f(x)$ (for $x \geq 0$) and **reflect it in the y-axis** to produce the left half. The original left half of the graph is discarded.

IB TIP

A reliable way to remember the difference: $|f(x)|$ affects the **output** (reflects y-values), so it looks like folding the graph up at the x-axis. $f(|x|)$ affects the **input** (replaces x with $|x|$), so the function receives non-negative inputs only, creating a symmetric graph about the y-axis.

⚠ **EXAM ALERT**

Paper 1 graph-sketching questions frequently ask you to sketch $y = |f(x)|$ given the graph of f. The two most common errors are: (1) reflecting the correct part of the graph the wrong way (should reflect the portion below the x-axis **upward**, not downward), and (2) leaving a corner point unlabelled at the x-axis where the reflection occurs.

Section 6: Polynomial Division, Factor & Remainder Theorems HL

This section covers AHL content for Topic 2. The factor and remainder theorems give you a systematic toolkit for factorising and evaluating polynomials — skills that appear in Paper 1 (factor theorem, quick remainder evaluation) and Paper 2 (full long division to find quotients).

6.1 Polynomial Long Division

When you divide a polynomial $p(x)$ by a divisor $d(x)$, you obtain a **quotient** $q(x)$ and a **remainder** $r(x)$, where the degree of $r(x)$ is strictly less than the degree of $d(x)$:

$$p(x) = d(x) \cdot q(x) + r(x)$$

For a linear divisor $(x - a)$, the remainder r is a constant.

Method — long division by $(x - a)$:

1. Write the dividend in descending powers, inserting 0 coefficients for any missing terms.
2. Divide the leading term of the current dividend by the leading term of the divisor. Write the result above the line as the next term of the quotient.
3. Multiply the divisor by that term and subtract.
4. Bring down the next term and repeat until the degree of the remainder is less than the degree of the divisor.

WORKED EXAMPLE

Polynomial long division

Divide $p(x) = x^3 - 6x^2 + 11x - 6$ *by* $(x-1)$.

Set up the division: $x^3 - 6x^2 + 11x - 6$ divided by $(x-1)$.

Step 1: Divide the leading term: $x^3 \div x = x^2$. Multiply: $x^2(x-1) = x^3 - x^2$. Subtract:

$(x^3 - 6x^2 + 11x - 6) - (x^3 - x^2) = -5x^2 + 11x - 6$

Step 2: Divide: $-5x^2 \div x = -5x$. Multiply: $-5x(x-1) = -5x^2 + 5x$. Subtract:

$(-5x^2 + 11x - 6) - (-5x^2 + 5x) = 6x - 6$

Step 3: Divide: $6x \div x = 6$. Multiply: $6(x-1) = 6x - 6$. Subtract:

$(6x - 6) - (6x - 6) = 0$

Result:

$x^3 - 6x^2 + 11x - 6 = (x-1)(x^2 - 5x + 6) + 0$

The remainder is 0, confirming that $(x-1)$ is a factor. The quotient $x^2 - 5x + 6 = (x-2)(x-3)$, so $p(x) = (x-1)(x-2)(x-3)$.

6.2 The Factor Theorem

MEMORISE THIS

Factor Theorem: $(x-a)$ is a factor of polynomial $p(x)$ if and only if $p(a) = 0$.

Equivalently: if $p(a) \neq 0$, then $(x-a)$ is **not** a factor of $p(x)$.

The factor theorem gives you a fast way to test whether a linear expression divides evenly into a polynomial, without performing long division. Once one factor is confirmed, long division (or synthetic division) yields the remaining quotient.

Rational Root Theorem (useful for finding candidates): If $p(x) = a_n x^n + \cdots + a_0$ has integer coefficients, any rational root has the form $\pm\dfrac{\text{factor of } a_0}{\text{factor of } a_n}$.

WORKED EXAMPLE

Finding all factors using the Factor Theorem

Factorise $p(x) = x^3 - 3x^2 - 4x + 12$ *completely.*

Step 1 — Test rational root candidates. By the rational root theorem, candidates are $\pm 1, \pm 2, \pm 3, \pm 6$.

$p(1) = 1 - 3 - 4 + 12 = 6 \neq 0$ $p(-1) = -1 - 3 + 4 + 12 = 12 \neq 0$ $p(2) = 8 - 12 - 8 + 12 = 0 \checkmark$

So $(x - 2)$ is a factor.

Step 2 — Divide out the known factor. Divide $p(x)$ by $(x - 2)$:

$x^3 - 3x^2 - 4x + 12 = (x - 2)(x^2 - x - 6)$

Verify: $(x - 2)(x^2 - x - 6)$. Expand: $x^3 - x^2 - 6x - 2x^2 + 2x + 12 = x^3 - 3x^2 - 4x + 12$. Correct.

Step 3 — Factorise the quadratic.

$x^2 - x - 6 = (x - 3)(x + 2)$

Result:

$p(x) = (x - 2)(x - 3)(x + 2)$

EXAM ALERT

In Paper 1, if you are asked to **show** that $(x - a)$ is a factor, use the Factor Theorem by computing $p(a) = 0$. A single line of substitution earns the method mark and the conclusion mark. Full polynomial long division is usually needed in Paper 2 or when the question asks you to find the remaining quotient after establishing a factor.

6.3 The Remainder Theorem

MEMORISE THIS

Remainder Theorem: When $p(x)$ is divided by $(x - a)$, the remainder equals $p(a)$.

More generally, when $p(x)$ is divided by $(ax + b)$, the remainder equals $p\left(-\frac{b}{a}\right)$.

The remainder theorem lets you find the remainder of a polynomial division without performing the full algorithm — just evaluate the polynomial at the appropriate value.

✎ WORKED EXAMPLE

Applying the Remainder Theorem

Find the remainder when $p(x) = 2x^3 - x^2 + 3x - 5$ *is divided by* $(x + 2)$.

The divisor is $(x + 2) = (x - (-2))$, so $a = -2$. Evaluate:

$p(-2) = 2(-2)^3 - (-2)^2 + 3(-2) - 5 = 2(-8) - 4 + (-6) - 5 = -16 - 4 - 6 - 5 = -31$

The remainder is -31. No long division required.

IB TIP

The remainder theorem is a Paper 1 time-saver. If the question asks only for the remainder (not the full quotient), substitute $x = a$ directly. Only perform long division when the quotient itself is needed.

6.4 Combining the Theorems — Solving Cubic Equations

The standard strategy for solving a cubic $p(x) = 0$:

1. **Find one root** by testing $\pm 1, \pm 2, \ldots$ (rational root candidates) using the factor theorem.
2. **Extract the linear factor** $(x - a)$, then divide $p(x)$ by $(x - a)$ to obtain a quadratic quotient.
3. **Solve the quadratic** by factorisation, completing the square, or the quadratic formula.

If $p(a) = 0$, then $(x - a)$ divides $p(x)$ exactly, and the quotient has degree one lower than $p(x)$.

✎ **WORKED EXAMPLE**

Solving a cubic equation completely

Solve $x^3 - 2x^2 - 5x + 6 = 0$.

Let $p(x) = x^3 - 2x^2 - 5x + 6$. Test rational root candidates $\pm 1, \pm 2, \pm 3, \pm 6$:

$p(1) = 1 - 2 - 5 + 6 = 0 \checkmark$

So $(x - 1)$ is a factor. Divide $p(x)$ by $(x - 1)$:

$x^3 - 2x^2 - 5x + 6 = (x - 1)(x^2 - x - 6)$

Factorise the quadratic:

$x^2 - x - 6 = (x - 3)(x + 2)$

Therefore:

$p(x) = (x - 1)(x - 3)(x + 2)$

Solutions: $x = 1, x = 3, x = -2$.

Check: $p(3) = 27 - 18 - 15 + 6 = 0$ and $p(-2) = -8 - 8 + 10 + 6 = 0$. Confirmed.

MCQ Practice

These questions are styled after IB AA HL Paper 1. Work each one before revealing the answer.

Section 1 MCQs — Language of Functions

Q1. The function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2 - 4$. Which of the following statements is correct?

A. f is one-to-one and its range equals its codomain.

B. f is many-to-one and its range equals its codomain.

C. f is one-to-one and its range is $[-4, \infty)$.

D. f is many-to-one and its range is $[-4, \infty)$.

Q2. The natural domain of $g(x) = \ln(2x - 6)$ is:

A. $x > 0$

B. $x > 3$

C. $x \geq 3$

D. $x \in \mathbb{R}$

Section 2 MCQs — Composite and Inverse Functions

Q3. Let $f(x) = 2x + 1$ and $g(x) = x^2 - 3$. Find $(g \circ f)(x)$.

A. $2x^2 - 5$

B. $4x^2 + 4x - 2$

C. $4x^2 + 4x + 1 - 3$, simplified to $4x^2 + 4x - 2$

D. $2x^2 - 5$ (same as A)

Q4. The function $f(x) = \dfrac{3x - 2}{x + 1}, x \neq -1$. Find $f^{-1}(2)$.

A. $x = 4$

B. $x = -4$

C. $x = 4$ (confirmed)

D. $x = \dfrac{1}{2}$

Section 3 MCQs — Transformations

Q5. The graph of $y = f(x)$ is translated by $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$. The equation of the resulting graph is:

A. $y = f(x - 3) + 2$

B. $y = f(x + 3) - 2$

C. $y = f(x + 3) + 2$

D. $y = f(x - 3) - 2$

Q6. The point $(4, -6)$ lies on the graph of $y = f(x)$. What point lies on the graph of $y = -f(2x)$?

A. $(2, 6)$

B. $(-4, 6)$

C. $(2, -6)$

D. $(8, 6)$

Section 4 MCQs — Rational Functions

Q7. The function $h(x) = \dfrac{x^2 - 1}{x^2 - x - 2}$. Which of the following correctly describes its behaviour?

A. One vertical asymptote at $x = 2$ and a hole at $x = -1$.

B. Vertical asymptotes at $x = 2$ and $x = -1$.

C. One vertical asymptote at $x = -1$ and a hole at $x = 2$.

D. No vertical asymptotes; a hole at $x = 1$.

Q8. As $x \to \infty$, the function $f(x) = \dfrac{3x^2 - 2x + 1}{6x^2 + x - 5}$ approaches:

A. 3

B. $\dfrac{1}{2}$

C. ∞

D. 0

Section 5 MCQs — Modulus and Inequalities

Q9. Solve $|5 - 2x| < 3$.

A. $x < 1$ or $x > 4$

B. $1 < x < 4$

C. $x < -4$ or $x > 1$

D. $-4 < x < 1$

Q10. Which of the following is the graph of $y = |x^2 - 4|$?

A. The parabola $y = x^2 - 4$ with the portion between $x = -2$ and $x = 2$ reflected above the x-axis.

B. The parabola $y = x^2 - 4$ with the portions outside $[-2, 2]$ reflected above the x-axis.

C. A parabola with vertex at $(0, 4)$ opening downward.

D. The graph of $y = x^2 - 4$ unchanged.

Section 6 MCQs — Polynomial Division, Factor & Remainder Theorems

Q11. Find the remainder when $p(x) = x^3 + 2x^2 - 3x + 4$ is divided by $(x + 1)$.

A. 0

B. 4

C. 8

D. -4

Q12. If $(x - 2)$ is a factor of $p(x) = x^3 - kx^2 + 3x - 2$, find the value of k.

A. $k = 1$

B. $k = 2$

C. $k = 3$

D. $k = 4$

IB TIP

Final exam strategy for Topic 2: In the 45 minutes you have for each Paper 1 section, rational function sketches and transformation questions together are worth roughly 15–20 marks. Always start with asymptotes and intercepts before drawing any curve. For modulus inequalities, write out both branches explicitly — never try to do them mentally. For composite and inverse functions, write the definitions out step-by-step; the mark scheme allocates method marks at each line.