Complex Numbers

IB HL Study Guide

Category 1: Cartesian Form, Arithmetic & the Argand Diagram

Every complex number has the form $z = a + bi$ where $a = \text{Re}(z)$ is the real part and $b = \text{Im}(z)$ is the imaginary part. The key rule is $i^2 = -1$. The **Argand diagram** is a 2D plane where the horizontal axis is the real axis and the vertical axis is the imaginary axis — every complex number corresponds to one unique point. Arithmetic follows ordinary algebra, substituting $i^2 = -1$ wherever it appears.

> **IB TIP**
>
> **Formula booklet entries for this section:** Modulus $|z| = \sqrt{a^2 + b^2}$ (memorise), conjugate $z^* = a - bi$ (memorise), $z \cdot z^* = |z|^2$ (memorise).

1.1 The Argand Diagram

Structure of the Argand Diagram

```
             Im(z)  (imaginary axis)
               |
          5  +   .  ← (-1, 5)  =  z = -1 + 5i
          4  |
          3  |
          2  |
          1  |
─────────────────────────────────── Re(z)  (real axis)
       -2 -1 | 1   2   3
             |
         -2  +   .  ← (3, -2)  =  z = 3 - 2i
             |
```

- $z = a + bi$ is plotted at the point (a, b)
- $\text{Re}(z) = a$ gives the horizontal axis position
- $\text{Im}(z) = b$ gives the vertical axis position

WORKED EXAMPLE

Review Set 14A — Q1

On an Argand diagram, illustrate the complex numbers: (a) $3 - 2i$ *(b)* $-1 + 5i$

Answer:

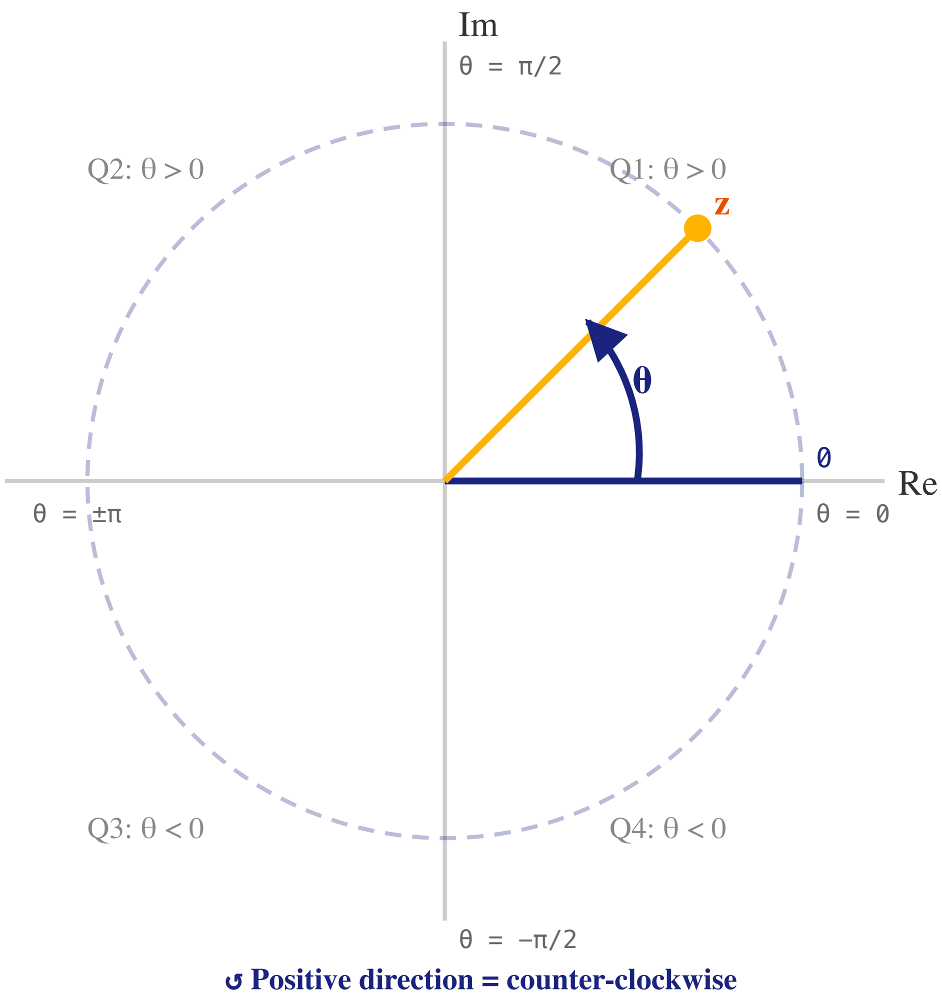
- $3 - 2i \to$ plot at $(3, -2)$: 3 right, 2 down [Quadrant 4]
- $-1 + 5i \to$ plot at $(-1, 5)$: 1 left, 5 up [Quadrant 2]

1.2 Modulus and Argument

The **modulus** $|z|$ is the distance from the origin to point z (Pythagoras). The **argument** $\arg(z)$ is the angle from the positive real axis to the line Oz, measured **counter-clockwise**. The **principal argument** is always in $(-\pi, \pi]$. You must **sketch the point first** — this reveals the quadrant, which determines which formula to apply.

MEMORISE THIS

Angle Convention — Counter-Clockwise from Positive Real Axis



Angles are always measured counter-clockwise from the positive real axis. Clockwise angles are negative.

- **Counter-clockwise** from the positive real axis = **positive** angle
- **Clockwise** from the positive real axis = **negative** angle
- The **principal argument** is always in the range $(-\pi, \pi]$
- Q1 and Q2 give positive arguments; Q3 and Q4 give negative arguments

Modulus and Argument — formulas

$|z| = \sqrt{a^2 + b^2} \qquad \text{[Pythagoras — always positive]}$

$\arg(z)$ — FIRST sketch the point to find its quadrant, THEN apply:

Quadrant / Position	Condition	Formula	Result range
Q1	$a > 0, b > 0$	$\arg = \arctan\left(\frac{b}{a}\right)$	$(0, \frac{\pi}{2})$
Q2	$a < 0, b > 0$	$\arg = \pi - \arctan!\left(\frac{	b
Q3	$a < 0, b < 0$	$\arg = -\pi + \arctan!\left(\frac{	b
Q4	$a > 0, b < 0$	$\arg = -\arctan!\left(\frac{	b
+Re axis	$b = 0, a > 0$	$\arg = 0$	
−Re axis	$b = 0, a < 0$	$\arg = \pi$	
+Im axis	$a = 0, b > 0$	$\arg = \frac{\pi}{2}$	
−Im axis	$a = 0, b < 0$	$\arg = -\frac{\pi}{2}$	

Quadrant map — which formula to use for $\arg(z)$

```
                  Im
                   |
   Quadrant 2      |       Quadrant 1
   a < 0, b > 0   |      a > 0, b > 0
   arg = pi − arctan(|b/a|) | arg = arctan(b/a)
                   |
   ────────────────┼─────────────────── Re
                   |
   Quadrant 3      |       Quadrant 4
   a < 0, b < 0   |      a > 0, b < 0
   arg = −pi+arctan(|b/a|) | arg = −arctan(|b/a|)
```

Key Steps:

1. Calculate $\left|\frac{b}{a}\right|$ — the ratio of absolute values
2. Take arctan of that ratio — this is the **reference angle** (always positive)
3. Apply the sign/shift above based on which quadrant your point is in

WORKED EXAMPLE

Review Set 14A — Q3

Find $|z|$ *and* $\arg z$ *for: (a)* $5 + 2i$ *(b)* $-5 + 2i$ *(c)* $-5 - 2i$ *(d)* $-2 - 5i$

Worked Solution — Modulus and argument, four cases

All four share the same modulus: $|z| = \sqrt{5^2 + 2^2} = \sqrt{25 + 4} = \sqrt{29}$

Step a: $z = 5 + 2i \to \text{Q1} \to \arg = \arctan\left(\frac{2}{5}\right) \approx 0.381$ rad (positive real, positive imag)

Step b: $z = -5 + 2i \to \text{Q2} \to \arg = \pi - \arctan\left(\frac{2}{5}\right) \approx 2.761$ rad (negative real, positive imag)

Step c: $z = -5 - 2i \to \text{Q3} \to \arg = -\pi + \arctan\left(\frac{2}{5}\right) \approx -2.761$ rad (negative real, negative imag)

Step d: $z = -2 - 5i \to \text{Q3} \to \arg = -\pi + \arctan\left(\frac{5}{2}\right) \approx -1.951$ rad (note $|b/a| = 5/2$, not $2/5$)

EXAM ALERT

Never just type $\arctan(b/a)$ into your calculator without sketching first. For $-5+2i$, the calculator gives a negative angle — but the argument is positive because the point is in Q2. The sketch takes 5 seconds and saves the mark.

1.3 Arithmetic — Addition, Multiplication, Division

Rules for complex arithmetic

Let $z = a + bi$ and $w = c + di$.

Operation	Formula
Addition	$z + w = (a+c) + (b+d)i$
Subtraction	$z - w = (a-c) + (b-d)i$
Multiplication	$z \cdot w = (ac - bd) + (ad + bc)i$ [expand brackets, use $i^2 = -1$]
Division	$\frac{z}{w} = \frac{z \cdot w^*}{

Conjugate: $z^* = a - bi$ [flip sign of imaginary part ONLY]

- $z + z^* = 2a$ (purely real)
- $z - z^* = 2bi$ (purely imaginary)
- $z \cdot z^* = a^2 + b^2 = |z|^2$ (real, ≥ 0)

Powers of i (cycle of 4): $i^1 = i, i^2 = -1, i^3 = -i, i^4 = 1, i^5 = i, \ldots$

To find i^n: divide n by 4, use the remainder: $r = 0 \rightarrow 1, r = 1 \rightarrow i, r = 2 \rightarrow -1, r = 3 \rightarrow -i$

WORKED EXAMPLE

Review Set 14A — Q2

$z = 3 + 2i, w = -2 + i$. *Find: (a)* $z + w$ *(b)* $2z - w$ *(c)* z^* *(d)* $3w^* - z^*$

Worked Solution — Arithmetic with two complex numbers

Step a: $z + w = (3-2) + (2+1)i = 1 + 3i$ (add real parts; add imag parts)

Step b: $2z - w = (6+4i) - (-2+i) = 8 + 3i$ (expand $2z = 6 + 4i$, then subtract)

Step c: $z^* = 3 - 2i$ (flip sign of imaginary part only)

Step d: $w^* = -2 - i \rightarrow 3w^* = -6 - 3i$ (conjugate flips $+i$ to $-i$)

$3w^* - z^* = (-6-3i) - (3-2i) = -9 - i$ (subtract term by term)

WORKED EXAMPLE

Review Set 14A — Q4

Given $|z| = 5$, *find: (a)* $|5i|$ *(b)* $|-4z^*|$ *(c)* $|(3+i)z|$ *(d)* $|i/z|$ *(e)* $|3/z|$ *(f)* $|(3+6i)/z|$

Worked Solution — Modulus algebra (rules: $|zw| = |z||w|$**,** $|z/w| = |z|/|w|$**,** $|z^*| = |z|$**)**

Core rules: $|z \cdot w| = |z| \cdot |w|, |z/w| = |z|/|w|, |z^*| = |z|, |i| = 1$

Step a: $|5i| = 5 \cdot |i| = 5 \cdot 1 = 5$ ($|5| = 5, |i| = 1$)

Step b: $|-4z^*| = 4 \cdot |z^*| = 4 \cdot 5 = 20$ ($|-4| = 4, |z^*| = |z| = 5$)

Step c: $|(3+i)z| = |3+i| \cdot |z| = \sqrt{10} \cdot 5 = 5\sqrt{10}$ ($|3+i| = \sqrt{9+1} = \sqrt{10}$)

Step d: $|i/z| = 1/5$ ($|i| = 1, |z| = 5$)

Step e: $|3/z| = 3/5$

Step f: $|(3+6i)/z| = \sqrt{45}/5 = \frac{3\sqrt{5}}{5}$ ($|3+6i| = \sqrt{9+36} = \sqrt{45} = 3\sqrt{5}$)

WORKED EXAMPLE

Review Set 14A — Q13

$z = 4 + i, w = 2 - 3i$. *Find: (a)* $2w^* - iz$ *(b)* $|w - z^*|$ *(c)* $|z^{100}|$ *(d)* $\arg(w - z)$

Worked Solution — Mixed operations including high power

$z = 4 + i, w = 2 - 3i, z^* = 4 - i, w^* = 2 + 3i$

Step a:

$iz = i(4+i) = 4i + i^2 = -1 + 4i$ (multiply out, $i^2 = -1$)

$2w^* = 2(2+3i) = 4 + 6i$

$2w^* - iz = (4+6i) - (-1+4i) = 5 + 2i$ (subtract)

Step b:

$w - z^* = (2-3i) - (4-i) = -2 - 2i$

$|w - z^*| = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$ (Pythagoras)

Step c:

$|z^{100}| = |z|^{100} = (\sqrt{17})^{100} = 17^{50}$ ($|z| = \sqrt{16+1} = \sqrt{17}$)

Step d:

$w - z = -2 - 4i \to$ Q3

$\arg(w - z) = -\pi + \arctan\left(\frac{4}{2}\right) = -\pi + \arctan(2) \approx -2.034$ rad (Q3 formula)

EXAM ALERT

Key modulus facts: $|z^*| = |z|$, $|iz| = |z|$, $|-z| = |z|$. All follow from $|z \cdot w| = |z||w|$ because $|i| = |-1| = 1$.

Category 2: Polar & Exponential Form

Any complex number can be written as $z = r(\cos\theta + i\sin\theta) = r\,\text{cis}\,\theta$ (polar form) or $z = re^{i\theta}$ (Euler/exponential form), where $r = |z|$ and $\theta = \arg(z)$. These forms make multiplication, division, and powers elegant: multiply moduli, add arguments. Converting accurately between Cartesian and polar — especially getting the quadrant right — is one of the highest-value skills in the entire topic.

IB TIP

Formula booklet entries for this section: Polar form $z = r\,\text{cis}\,\theta$ (given), Euler form $z = re^{i\theta}$ (given), polar multiplication/division (given).

2.1 Converting Between Forms

All three forms and conversion rules

$z = a + bi = r(\cos\theta + i\sin\theta) = r\,\text{cis}\,\theta = r \cdot e^{i\theta}$

Direction	Formulas
Cartesian → Polar	$r = \sqrt{a^2 + b^2}, \theta = \arg(z)$ [use quadrant rules from Cat 1]
Polar → Cartesian	$a = r\cos\theta, b = r\sin\theta$

Key polar operations:

Operation	Formula	Rule
Multiply	$z_1 \cdot z_2 = r_1 r_2 \cdot e^{i(\theta_1+\theta_2)}$	moduli MULTIPLY, arguments ADD
Divide	$\frac{z_1}{z_2} = \frac{r_1}{r_2} \cdot e^{i(\theta_1-\theta_2)}$	moduli DIVIDE, arguments SUBTRACT
Power	$z^n = r^n \cdot e^{in\theta}$	modulus to power n, argument $\times n$
Conjugate	$z^* = r \cdot e^{-i\theta}$	same modulus, NEGATE argument

Geometric meaning of multiplication by $r \cdot e^{i\varphi}$:

- ROTATE by angle φ AND SCALE distance from origin by factor r
- Multiply by $i = e^{i\pi/2}$: rotate 90° anticlockwise, no scaling

The triangle connecting Cartesian and polar

```
               Im
                |  /  ← z = a + bi = r·e^(iθ)
              b +./
                /   ↑ b = r·sinθ
             r /    |
              / θ   |
  ──────────────────────────────  Re
```

```
                a
                ←→
        a = r·cosθ
```

Conversion steps (Cartesian to Polar):

1. Draw the point (a, b) — identify the quadrant
2. Compute $r = \sqrt{a^2 + b^2}$
3. Compute reference angle $= \arctan\left(\frac{|b|}{|a|}\right)$
4. Adjust sign/shift for the quadrant (see Category 1)
5. Write $z = r \cdot e^{i\theta}$ or $z = r \operatorname{cis} \theta$

WORKED EXAMPLE

Worked Conversion: $-1 + i\sqrt{3} \to$ **Polar**

Step 1: $a = -1, b = \sqrt{3} \to$ sketch shows Q2 ($a < 0, b > 0$) — identify quadrant

Step 2: $r = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$ — modulus by Pythagoras

Step 3: Reference angle $= \arctan\left(\frac{\sqrt{3}}{1}\right) = \arctan(\sqrt{3}) = \frac{\pi}{3}$ — arctan of $|b/a|$

Step 4: Q2 formula: $\arg = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ — apply quadrant shift

Step 5: Answer: $z = 2 \cdot e^{i \cdot 2\pi/3} = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ — write polar form

WORKED EXAMPLE

Review Set 14A — Q11

Write $-1 + i\sqrt{3}$ in polar form. Hence find all values of n for which $(-1 + i\sqrt{3})^n$ is real.

Worked Solution — Polar form then condition for real power

Polar form (from conversion above): $z = 2 \cdot e^{i \cdot 2\pi/3}$

Step 1: $z^n = 2^n \cdot e^{i \cdot n \cdot 2\pi/3} = 2^n\left(\cos\frac{2\pi n}{3} + i\sin\frac{2\pi n}{3}\right)$ — de Moivre

Step 2: For z^n to be REAL: imaginary part $= 0$, so $\sin\left(\frac{2\pi n}{3}\right) = 0$ — condition for real

Step 3: $\sin(k\pi) = 0$ for any integer k, so $\frac{2\pi n}{3} = k\pi \to n = \frac{3k}{2}$ — solve for n

Step 4: For INTEGER n: $\frac{3k}{2}$ is integer only when k is even ($k = 2m$)

Step 5: $n = 3m$ for integer m. **Answer:** n **is any multiple of 3**

WORKED EXAMPLE

Review Set 14A — Q15

Write $2 - 2\sqrt{3}\,i$ *in polar form. Hence find all values of* n *for which* $(2 - 2\sqrt{3}\,i)^n$ *is purely imaginary.*

Worked Solution — Polar form then condition for purely imaginary

Step 1: $r = \sqrt{4 + 12} = \sqrt{16} = 4$. Point $(2, -2\sqrt{3})$ is Q4.

Step 2: $\arg = -\arctan\left(\frac{2\sqrt{3}}{2}\right) = -\arctan(\sqrt{3}) = -\frac{\pi}{3}$ (Q4: negate arctan)

Step 3: Polar: $z = 4 \cdot e^{-i\pi/3} = 4\,\text{cis}\left(-\frac{\pi}{3}\right)$

Step 4: $z^n = 4^n \cdot \text{cis}\left(-\frac{n\pi}{3}\right)$. For purely imaginary: $\text{Re}(z^n) = 0$ — condition

Step 5: $\cos\left(-\frac{n\pi}{3}\right) = 0 \to \frac{n\pi}{3} = \frac{\pi}{2} + k\pi \to n = \frac{3}{2} + 3k$ — solve

Step 6: Not an integer for any k. Check period: arguments cycle every 6 steps.

Step 7: $n = 1$: $4\,\text{cis}(-\pi/3)$, $n = 2$: $16\,\text{cis}(-2\pi/3)$, $n = 3$: $64\,\text{cis}(-\pi) = -64$ [real, not imag] — direct check

Step 8: No positive integer n gives a purely imaginary result for this z.

WORKED EXAMPLE

Review Set 14A — Q17

$z = 4\,\text{cis}\,\theta$. *Find the modulus and argument of: (a)* z^3 *(b)* $1/z$ *(c)* iz^n

Worked Solution — Polar operations on $z = 4\,\text{cis}\,\theta$

Given: $|z| = 4, \arg(z) = \theta$. Apply polar rules:

Step a: z^3: $|z^3| = 4^3 = 64, \arg(z^3) = 3\theta$ (modulus cubed, arg tripled)

Step b: $1/z$: $|1/z| = 1/4, \arg(1/z) = -\theta$ (negate argument for reciprocal)

Step c: iz^n: $|i| = 1, \arg(i) = \pi/2$

$|iz^n| = 1 \cdot 4^n = 4^n, \arg(iz^n) = \frac{\pi}{2} + n\theta$ (add arguments)

WORKED EXAMPLE

Review Set 14A — Q19/20

Use polar form to show $|1/z| = 1/|z|$ *and* $\arg(1/z) = -\arg z$. *Write* $1+i$ *in polar form.*

Worked Solution — Polar proof and conversion

Step 1: Let $z = r \cdot e^{i\theta}$. Then $1/z = (1/r) \cdot e^{-i\theta}$ — reciprocal in polar

Step 2: $|1/z| = 1/r = 1/|z|$ ✓ — modulus of $(1/r)e^{-i\theta}$ is $1/r$

Step 3: $\arg(1/z) = -\theta = -\arg(z)$ ✓ — argument is $-\theta$

Step 4: $1+i$: $r = \sqrt{2}$, Q1: $\arg = \arctan(1/1) = \pi/4$ — conversion

Step 5: $1+i = \sqrt{2} \cdot e^{i\pi/4} = \sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)$ — polar form

EXAM ALERT

Verify any polar conversion by substituting back: if $z = r\,\text{cis}\,\theta$, then $r\cos\theta$ should give a and $r\sin\theta$ should give b. A 10-second check catches errors before they lose marks.

Category 3: De Moivre's Theorem & Powers

De Moivre's theorem states $(r\,\text{cis}\,\theta)^n = r^n\,\text{cis}(n\theta)$. It works for all integer n (and rational n for roots). It has two key uses: (1) computing high powers of complex numbers quickly, and (2) proving trigonometric identities by expanding $(\cos\theta + i\sin\theta)^n$ in two ways. The $z + 1/z$ **technique** — letting $z = e^{i\theta}$ on the unit circle — is the standard approach for expressing $\cos^n\theta$ or $\sin^n\theta$ in terms of multiple angles, enabling powerful integrations.

IB TIP

Formula booklet entries for this section: De Moivre's theorem $(r\,\text{cis}\,\theta)^n = r^n\,\text{cis}(n\theta)$ (given), $z + 1/z = 2\cos\theta$ (memorise).

3.1 De Moivre's Theorem — Statement and Direct Use

De Moivre's Theorem — all three equivalent forms

$(r\,\text{cis}\,\theta)^n = r^n\,\text{cis}(n\theta)$ for $n \in \mathbb{Z}$ (and $n \in \mathbb{Q}$ for roots)

$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$ [special case: $r = 1$]

$(r \cdot e^{i\theta})^n = r^n \cdot e^{in\theta}$ [Euler form]

What de Moivre does geometrically:

- Raise to power $n \to$ scale modulus from r to r^n, multiply argument by n
- $|z^n| = |z|^n$ and $\arg(z^n) = n \cdot \arg(z)$ [adjust to $(-\pi, \pi]$ if needed]

WORKED EXAMPLE

Review Set 14A — Q6

If $(x+iy)^n = X + Yi$ *where* n *is a positive integer, show that* $X^2 + Y^2 = (x^2+y^2)^n$.

Worked Solution — De Moivre modulus proof

Step 1: Write $z = x + iy$ in polar form: $z = r \cdot e^{i\theta}$ where $r = \sqrt{x^2+y^2}$ — convert to polar

Step 2: By de Moivre: $z^n = r^n \cdot e^{in\theta} = X + Yi$ — apply theorem

Step 3: Read off Cartesian parts: $X = r^n \cos(n\theta), Y = r^n \sin(n\theta)$ — real and imag parts

Step 4: $X^2 + Y^2 = r^{2n}\cos^2(n\theta) + r^{2n}\sin^2(n\theta) = r^{2n}(\cos^2(n\theta) + \sin^2(n\theta))$ — sum of squares

Step 5: $= r^{2n} \cdot 1 = r^{2n} = \left(\sqrt{x^2+y^2}\right)^{2n} = (x^2+y^2)^n$ ✓ — substitute r, done

WORKED EXAMPLE

Review Set 14A — Q10

$z = 4\,\text{cis}\,\theta$. *Find modulus and argument of: (a)* $(2z)^{-1}$ *(b)* $1 - z$

Worked Solution — Combined polar operations

Step a: $(2z)^{-1}$: $|2z| = 2 \cdot 4 = 8, \arg(2z) = \theta$ ($\times 2$ is real, adds 0 to arg)

$|(2z)^{-1}| = \frac{1}{8}, \arg((2z)^{-1}) = -\theta$ — negate argument for reciprocal

Step b: $z = 4\cos\theta + 4i\sin\theta, 1 - z = (1 - 4\cos\theta) - 4i\sin\theta$ — convert to Cartesian then subtract

$|1 - z| = \sqrt{(1-4\cos\theta)^2 + 16\sin^2\theta}$ — Pythagoras

$\arg(1-z) = \arctan\left(\frac{-4\sin\theta}{1-4\cos\theta}\right)$ [use correct quadrant] — argument

3.2 Trig Identities — The $z + 1/z$ Technique

Let $z = e^{i\theta}$ (unit circle). Then $z + 1/z = 2\cos\theta$ and $z - 1/z = 2i\sin\theta$. Raising these to powers and expanding with the binomial theorem gives identities for $\cos^n\theta$ and $\sin^n\theta$ as combinations of $\cos(k\theta)$ — essential for integration and proof questions.

The $z + 1/z$ technique — key identities

Let $z = e^{i\theta}$, so $z = \cos\theta + i\sin\theta$ and $|z| = 1$.

$z + z^{-1} = 2\cos\theta \qquad$ [de Moivre $n = 1$ and $n = -1$, add]

$z - z^{-1} = 2i\sin\theta \qquad$ [de Moivre $n = 1$ and $n = -1$, subtract]

$z^n + z^{-n} = 2\cos(n\theta)$ [generalised — use this to GROUP pairs]

$z^n - z^{-n} = 2i\sin(n\theta)$ [generalised]

Strategy: $(z + z^{-1})^n = (2\cos\theta)^n = 2^n \cos^n\theta$

- Expand left side by binomial, group $z^k + z^{-k}$ pairs $\to 2\cos(k\theta)$
- This converts $\cos^n\theta$ into a sum of cosines of multiples of θ

How the $z + 1/z$ technique works — visual flow

```
GOAL: express cos^5(θ) in terms of cos(5θ), cos(3θ), cos(θ)

(z + 1/z)^5  ═══[equals]═══  (2cosθ)^5 = 32cos^5(θ)
       ║
       ║  expand by binomial theorem  (coefficients: 1  5  10
       ↓
z^5 + 5z^3 + 10z + 10z^(−1) + 5z^(−3) + z^(−5)
       ║
       ║  group symmetric pairs: (z^5+z^(−5)) + 5(z^3+z^(−3)) +
       ↓
2cos(5θ) + 5·2cos(3θ) + 10·2cos(θ)  =  2cos5θ + 10cos3θ + 20cos

Equate: 32cos^5(θ) = 2cos5θ + 10cos3θ + 20cosθ

Divide:  cos^5(θ) = (1/16)(cos5θ + 5cos3θ + 10cosθ)   ✓
```

WORKED EXAMPLE

Oxford Key — Q14a

Express $\cos^5\theta$ *in terms of* $\cos(n\theta)$ *using the* $z + 1/z$ *technique*.

Worked Solution — Full expansion step by step

Step 1: Set $z = e^{i\theta}$. Then $(z + z^{-1})^5 = (2\cos\theta)^5 = 32\cos^5\theta$ — setup

Step 2: Expand $(z + z^{-1})^5$ by binomial theorem (coefficients $1, 5, 10, 10, 5, 1$):

$= z^5 + 5z^3 + 10z + 10z^{-1} + 5z^{-3} + z^{-5}$

Step 3: Group symmetric pairs:

$= (z^5 + z^{-5}) + 5(z^3 + z^{-3}) + 10(z + z^{-1})$

Step 4: Apply $z^n + z^{-n} = 2\cos(n\theta)$ to each group:

$= 2\cos(5\theta) + 5\cdot 2\cos(3\theta) + 10\cdot 2\cos(\theta) = 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta$

Step 5: Equate: $32\cos^5\theta = 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta$ — set equal to right side

Step 6: Divide by 32:

$\cos^5\theta = \frac{1}{16}\left[\cos 5\theta + 5\cos 3\theta + 10\cos\theta\right]$ ✓

WORKED EXAMPLE

Oxford Key — Q14b

Hence evaluate $\displaystyle\int \cos^5\theta\, d\theta$.

Worked Solution — Integration using the multiple-angle form

$\cos^5\theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$ [from Q14a above]

$\int \cos^5\theta\, d\theta = \frac{1}{16}\int(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)\, d\theta$

Step 1: $\displaystyle\int \cos 5\theta\, d\theta = \frac{\sin 5\theta}{5}$

Step 2: $\displaystyle\int 5\cos 3\theta\, d\theta = \frac{5\sin 3\theta}{3}$ (coefficient 5 stays)

Step 3: $\displaystyle\int 10\cos\theta\, d\theta = 10\sin\theta$

Step 4: Combine:

$\frac{1}{16}\left[\frac{\sin 5\theta}{5} + \frac{5\sin 3\theta}{3} + 10\sin\theta\right] + C$

Step 5: Simplify:

$\frac{\sin 5\theta}{80} + \frac{5\sin 3\theta}{48} + \frac{5\sin\theta}{8} + C$

(verified by differentiation ✓)

EXAM ALERT

Critical error to avoid: writing $\frac{\sin 3\theta}{16}$ for the middle term. The correct term is $\frac{1}{16} \times \frac{5\sin 3\theta}{3} = \frac{5\sin 3\theta}{48}$. The binomial coefficient 5 does NOT disappear — it stays in the numerator.

WORKED EXAMPLE

Oxford Key — Trig Identity

Use de Moivre's theorem to prove $\cos(2\theta) = \cos^2\theta - \sin^2\theta$ *and* $\sin(2\theta) = 2\sin\theta\cos\theta$.

Worked Solution — Double angle identities from de Moivre

Step 1: $(\cos\theta + i\sin\theta)^2 = \cos(2\theta) + i\sin(2\theta)$ [de Moivre, $n = 2$] — apply theorem

Step 2: Expand left side:

$(\cos\theta + i\sin\theta)^2 = \cos^2\theta + 2i\cos\theta\sin\theta + i^2\sin^2\theta = \cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta$

Step 3: Equate real parts: $\cos(2\theta) = \cos^2\theta - \sin^2\theta$ ✓

Step 4: Equate imaginary parts: $\sin(2\theta) = 2\sin\theta\cos\theta$ ✓

EXAM ALERT

For any n, this approach gives $\cos(n\theta)$ and $\sin(n\theta)$ as polynomials in $\cos\theta$ and $\sin\theta$. For $n = 3$: expand $(\cos\theta + i\sin\theta)^3$ using binomial $(1, 3, 3, 1)$, then equate real and imaginary parts.

Category 4: nth Roots of Complex Numbers & Roots of Unity

The equation $z^n = w$ always has exactly n **solutions** in $\mathbb{C}$. These are called the **nth roots** of w. Using de Moivre in reverse gives a single formula for all of them. The roots are equally spaced at angles of $\frac{2\pi}{n}$ around a circle of radius $|w|^{1/n}$ — they form a regular n-gon on the Argand diagram. The **nth roots of unity** (solutions of $z^n = 1$) always sum to zero, which is a key algebraic identity used in many exam proofs.

IB TIP

Formula booklet entries for this section: nth roots formula (given), sum of nth roots of unity = 0 (memorise).

4.1 The nth Root Formula

Finding all nth roots of $w = r \cdot e^{i\theta}$

$$w^{1/n} = r^{1/n} \cdot \text{cis}\left(\frac{\theta + 2k\pi}{n}\right) \qquad \text{for } k = 0, 1, 2, \ldots, n-1$$

For each root:

- **Modulus:** $r^{1/n}$ [same for ALL n roots]
- **Argument:** $\frac{\theta + 2\pi k}{n}$ [increases by $\frac{2\pi}{n}$ for each step in k]

Geometry: all n roots lie on a circle of radius $r^{1/n}$. They are equally spaced by $\frac{2\pi}{n} = \frac{360^\circ}{n}$. The n roots form a regular n-gon on the Argand diagram.

Geometry of cube roots — regular triangle inscribed on circle

```
               Im
                |    * root k=1  (angle θ/3 + 2π/3)
                |  ↗
         r^(1/3)|
                O────────────────── Re
               /|  → root k=0  (angle θ/3)
              ↙ |
  root k=2     |
  (angle θ/3 + 4π/3)

  ● All three on a circle of radius r^(1/3)
  ● Each pair separated by 2π/3 = 120°
```

```
● They form an equilateral triangle
General rule: nth roots form a regular n-gon, spacing = 2π/n
```

Step-by-step process for finding nth roots:

1. Write the target number w in polar form: $w = r \cdot e^{i\theta}$. Use quadrant rules for θ.
2. Compute the modulus of each root: $r^{1/n}$
3. Compute the argument for each $k = 0, 1, \ldots, n-1$: $\theta_k = \frac{\theta + 2\pi k}{n}$
4. Convert each root to Cartesian form if needed: $a_k = r^{1/n}\cos(\theta_k), b_k = r^{1/n}\sin(\theta_k)$
5. Verify: cube each root (or raise to power n) — all should give back w

WORKED EXAMPLE

Review Set 14A — Q9

Find the cube roots of $-64i$, giving answers in Cartesian form.

Worked Solution — Cube roots of $-64i$

Step 1: $-64i$ lies on the negative imaginary axis: $r = 64, \arg = -\frac{\pi}{2}$ — polar form

Step 2: Write: $-64i = 64 \cdot e^{i(-\pi/2)}$

Step 3: Modulus of each root: $64^{1/3} = 4$ — cube root of 64

Step 4: Arguments: $\theta_k = \frac{-\pi/2 + 2\pi k}{3}$ for $k = 0, 1, 2$ — apply formula

- $k = 0$: $\theta = -\frac{\pi}{6}$
- $k = 1$: $\theta = -\frac{\pi}{6} + \frac{2\pi}{3} = \frac{\pi}{2}$
- $k = 2$: $\theta = -\frac{\pi}{6} + \frac{4\pi}{3} = \frac{7\pi}{6} \to$ use $-\frac{5\pi}{6}$ (adjust to $(-\pi, \pi]$)

Step 5: Convert to Cartesian:

- $k = 0$: $4 \cdot \text{cis}\left(-\frac{\pi}{6}\right) = 4\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right) = 2\sqrt{3} - 2i$
- $k = 1$: $4 \cdot \text{cis}\left(\frac{\pi}{2}\right) = 4(0 + i) = 4i$
- $k = 2$: $4 \cdot \text{cis}\left(-\frac{5\pi}{6}\right) = 4\left(-\frac{\sqrt{3}}{2} - \frac{i}{2}\right) = -2\sqrt{3} - 2i$

Step 6: Verify: $(2\sqrt{3} - 2i)^3 = -64i$ ✓ (check by expansion)

WORKED EXAMPLE

Review Set 14A — Q16

Determine the cube roots of -27.

Worked Solution — Cube roots of -27

Step 1: -27 is on the negative real axis: $r = 27, \arg = \pi$

Step 2: Write: $-27 = 27 \cdot e^{i\pi}$

Step 3: Modulus of each root: $27^{1/3} = 3$

Step 4: Arguments: $\theta_k = \frac{\pi+2\pi k}{3}$ for $k = 0, 1, 2$

- $k = 0$: $\theta = \frac{\pi}{3} \to 3 \cdot \text{cis}\left(\frac{\pi}{3}\right) = 3\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) = \frac{3}{2} + \frac{3\sqrt{3}}{2}\,i$
- $k = 1$: $\theta = \pi \to 3 \cdot \text{cis}(\pi) = 3(-1+0) = -3$
- $k = 2$: $\theta = \frac{5\pi}{3} \to 3 \cdot \text{cis}\left(\frac{5\pi}{3}\right) = 3\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) = \frac{3}{2} - \frac{3\sqrt{3}}{2}\,i$

4.2 Roots of Unity — Definitions and Key Identity

The **nth roots of unity** are the n solutions to $z^n = 1$. They can all be written as ω^k where $\omega = e^{2\pi i/n}$. A critical algebraic fact: **the sum of all nth roots of unity equals zero**. This follows from Vieta's formulas on $z^n - 1 = 0$. For **cube roots of unity**, the non-real root ω satisfies $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$ — these two identities are used to simplify almost every expression involving ω.

Roots of unity — all key identities

nth roots of unity: $z_k = e^{2\pi i k/n}$ for $k = 0, 1, \ldots, n-1$

- They lie equally spaced on the UNIT CIRCLE, forming a regular n-gon
- Sum of ALL nth roots of unity $= 0$ [Vieta on $z^n - 1 = 0$]

Cube roots of unity ($\omega^3 = 1, \omega \neq 1$):

$\omega = e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}\,i$

Identity	Notes
$1 + \omega + \omega^2 = 0$	$\to \omega + \omega^2 = -1$ — USE THIS CONSTANTLY
$\omega \cdot \omega^2 = \omega^3 = 1$	product of roots
$\omega^2 = \omega^*$	they are complex conjugates of each other
$\omega^2 = 1/\omega$	since $\omega^3 = 1 \to \omega^2 = \omega^{-1}$
Powers cycle: $\omega^3 = 1, \omega^4 = \omega, \omega^5 = \omega^2$, $\omega^6 = 1, \ldots$	period 3

WORKED EXAMPLE

Review Set 14A — Q22

State the five fifth roots of unity and hence solve: (a) $(2z-1)^5 = 32$ *(b)* $z^5+5z^4+10z^3+10z^2+5z=0$ *(c)* $(z+1)^5=(z-1)^5$

Worked Solution — Fifth roots of unity and applications

Fifth roots of unity: $z_k = e^{2\pi ik/5}$ for $k=0,1,2,3,4$

$= 1,\ e^{2\pi i/5},\ e^{4\pi i/5},\ e^{6\pi i/5},\ e^{8\pi i/5}$

Their sum $=0$ (Vieta's formulas on $z^5-1=0$) ✓

Step a: $(2z-1)^5 = 32 = 2^5\cdot 1 \to (2z-1)^5 = 2^5$

$2z-1 = 2\cdot z_k \to z = \frac{1+2\cdot e^{2\pi ik/5}}{2}$ for $k=0,1,2,3,4$ — 5 solutions

Step b: $z^5+5z^4+10z^3+10z^2+5z=0 \to z(z^4+5z^3+10z^2+10z+5)=0$ — factor out z

Remaining factor: note this is related to the $(z+1)^5$ expansion.

$z=0$ is one solution; remaining 4 from $z^4+5z^3+\cdots=0$

Step c: $(z+1)^5=(z-1)^5 \to \left(\frac{z+1}{z-1}\right)^5 = 1$ [divide both sides] — $z\neq 1$

Let $w=\frac{z+1}{z-1}$. Then $w^5=1$, so $w=e^{2\pi ik/5}$ for $k=0,1,2,3,4$

$w=1$ ($k=0$) gives $\frac{z+1}{z-1}=1 \to$ no solution. Use $k=1,2,3,4$.

For each k: $z=\frac{w+1}{w-1}$ where $w=e^{2\pi ik/5}$ — 4 solutions

WORKED EXAMPLE

Oxford Key — Exam-Style

$(1+i)^{10}$ *— find its value. (Using de Moivre)*

Worked Solution — $(1+i)^{10}$ using de Moivre

Step 1: Write $1+i$ in polar: $r=\sqrt{2}$, $\arg = \frac{\pi}{4} \to 1+i=\sqrt{2}\cdot e^{i\pi/4}$

Step 2: $(1+i)^{10} = (\sqrt{2})^{10}\cdot e^{i\cdot 10\pi/4}$ by de Moivre

Step 3: $(\sqrt{2})^{10} = 2^5 = 32$

Step 4: $\frac{10\pi}{4} = \frac{5\pi}{2} = 2\pi+\frac{\pi}{2} \to$ equivalent angle $=\frac{\pi}{2}$ — reduce to $(-\pi,\pi]$

Step 5: $=32\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right) = 32(0+i) = 32i$ — **final answer**

EXAM ALERT

The sum of all nth roots of unity $= 0$. For IB proofs: factor $z^n - 1 = (z-1)(z^{n-1} + \cdots + z + 1) = 0$, so the sum of ALL non-trivial roots equals -1, and including $z = 1$ gives sum $= 0$.

Category 5: Loci & Regions in the Complex Plane

A **locus** is the set of all complex numbers z satisfying a given geometric condition. The standard approach: write $z = x + iy$, translate the condition into an equation in x and y, then simplify to recognise a known curve. There are five standard loci; knowing their geometric meaning immediately tells you the shape before you start the algebra.

IB TIP

Formula booklet entries for this section: All locus formulas must be memorised — not in the formula booklet.

5.1 The Five Standard Loci

Condition	Locus (Geometric Shape) + Method
$\|z - a\| = r$	**Circle**, centre a, radius r. Method: $\|z - (p + qi)\| = r$ means the point z is distance r from (p, q). Example: $\|z - (2 + 3i)\| = 5 \to$ circle centred $(2, 3)$, radius 5.
$\|z - a\| = \|z - b\|$	**Perpendicular bisector** of the segment joining a to b. Meaning: z is equidistant from the two fixed points a and b. Method: set $\|\|x + iy - a\|^2 = \|\|x + iy - b\|^2$, expand, simplify.
$\arg(z - a) = \theta$	**Ray** starting at point a (but NOT including a), making angle θ with the positive real direction. Always state the constraint (e.g. $x > \text{Re}(a)$) to specify which ray.
$\|z - a\| \leq r$	**Closed disc** — circle plus its interior. Draw a solid boundary. $<$ means open disc — dashed boundary (point not included).
$\text{Re}(z) = k$ or $\text{Im}(z) = k$	**Vertical line** $x = k$, or **horizontal line** $y = k$ on the Argand diagram.

5.2 Worked Locus Examples

How to derive a locus — the general method

```
GIVEN: a condition on z  (e.g. |z−i| = |z−2| or arg(z−1) = π/3)

STEP 1: Write z = x + iy
STEP 2: Substitute into the condition
STEP 3: Expand and simplify the equation in x and y
STEP 4: Identify the shape:
           x² + y² + ... = r²  →  circle
```

```
        ax + by = c              →  straight line
        y = mx + c               →  ray (with direction constrai
```

WORKED EXAMPLE

Review Set 14A — Q14

Find the Cartesian equation of the locus of $P(x, y)$ *where* $z = x + iy$ *and: (a)* $\arg(z - i) = \frac{\pi}{6}$ *(b)* $\left|\frac{z+2}{z-2}\right| = 2$

Worked Solution — Two loci derived from scratch

PART (a): $\arg(z - i) = \frac{\pi}{6}$

Step 1: $z - i = x + (y - 1)i$ — substitute $z = x + iy$

Step 2: $\arg = \frac{\pi}{6}$ means $\tan\left(\frac{\pi}{6}\right) = \frac{\text{Im}}{\text{Re}} = \frac{y-1}{x}$ — tan of argument

Step 3: $\frac{1}{\sqrt{3}} = \frac{y-1}{x} \to x = \sqrt{3}(y - 1) \to y = \frac{x}{\sqrt{3}} + 1$ — solve for y

Step 4: This is a **RAY** at angle $\frac{\pi}{6}$, starting from $(0, 1)$, going into $x > 0$ — add constraint

PART (b): $\left|\frac{z+2}{z-2}\right| = 2$

Step 1: $|z + 2| = 2|z - 2| \to |z + 2|^2 = 4|z - 2|^2$ — square both sides

Step 2: $(x + 2)^2 + y^2 = 4[(x - 2)^2 + y^2]$ — expand moduli

Step 3: $x^2 + 4x + 4 + y^2 = 4x^2 - 16x + 16 + 4y^2$ — expand brackets

Step 4: $0 = 3x^2 - 20x + 12 + 3y^2$ — rearrange

Step 5: $x^2 - \frac{20}{3}x + 4 + y^2 = 0 \to \left(x - \frac{10}{3}\right)^2 + y^2 = \frac{100}{9} - \frac{36}{9} = \frac{64}{9}$ — complete the square

Step 6: Circle: centre $\left(\frac{10}{3}, 0\right)$, radius $= \frac{8}{3}$ — final answer

EXAM ALERT

For part (a): $\arg(z - a) = \theta$ is a RAY, not a full line. You must state the direction constraint (here $x > 0$) to eliminate the opposite ray. Omitting this constraint loses a mark.

WORKED EXAMPLE

Review Set 14A — Q18

Illustrate the region defined by: $\{z \mid 2 \leq |z| \leq 5 \text{ and } -\frac{\pi}{4} < \arg z \leq \frac{\pi}{2}\}$. *Show all included boundary points.*

Worked Solution — Region defined by modulus AND argument constraints

Condition 1: $2 \leq |z| \leq 5 \to$ the region BETWEEN two circles

- $|z| = 2$: inner circle (solid boundary — included because $\geq$)
- $|z| = 5$: outer circle (solid boundary — included because $\leq$)

Condition 2: $-\frac{\pi}{4} < \arg(z) \leq \frac{\pi}{2} \to$ a sector (wedge)

- $\arg = -\frac{\pi}{4}$: lower boundary (DASHED — not included because $<$)
- $\arg = \frac{\pi}{2}$: upper boundary (SOLID — included because $\leq$)

Combined region: a sector of an annulus (ring)

Draw the two arcs and two radii, mark which boundaries are solid vs dashed.

WORKED EXAMPLE

Review Set 14A — Q21

$P_1P_2P_3$ *is an isosceles triangle where* $P_1P_3 = \sqrt{3} \cdot P_1P_2$. O *is origin.* $OP_1 = z_1$, $OP_2 = z_2$, $OP_3 = z_3$. $\arg(z_3 - z_1) = \alpha$. *(a) Show* $\arg(z_3 - z_2) = \alpha - \frac{\pi}{2}$ *(b) Find modulus and argument of* $\frac{z_3 - z_1}{z_3 - z_2}$

Worked Solution — Complex numbers as position vectors in geometry

Step 1: $z_3 - z_1$ and $z_3 - z_2$ are the vectors $\overrightarrow{P_1P_3}$ and $\overrightarrow{P_2P_3}$ respectively — interpret geometrically

Step 2: $P_1P_3 = \sqrt{3} \cdot P_1P_2$ means $|z_3 - z_1| = \sqrt{3}\,|z_3 - z_2|$ — given length condition

Step 3: $P_1P_2P_3$ isosceles: angle at $P_3 = 90^\circ$ (since $P_1P_3 = \sqrt{3} \cdot P_1P_2$ and isosceles) — geometry

Step 4: $\arg(z_3 - z_2) = \arg(z_3 - z_1) - \frac{\pi}{2} = \alpha - \frac{\pi}{2}$ — perpendicularity gives $-\pi/2$

Step 5: $\left|\frac{z_3 - z_1}{z_3 - z_2}\right| = \sqrt{3}$, $\arg = \alpha - \left(\alpha - \frac{\pi}{2}\right) = \frac{\pi}{2}$ — divide the complex numbers

So $\frac{z_3 - z_1}{z_3 - z_2} = \sqrt{3} \cdot e^{i\pi/2} = i\sqrt{3}$ — Cartesian form

EXAM ALERT

A key technique: if you need to find the angle between two lines on the Argand diagram, DIVIDE the corresponding complex number differences. The argument of z_1/z_2 equals $\arg(z_1) - \arg(z_2)$. This is equivalent to finding the angle at a vertex of a triangle.

Category 6: Applications, Proofs & Exam-Style Problems

This category covers multi-part IB exam questions that combine several earlier techniques. The **IB Problem** document contains two full exam questions (18 and 19 marks). These questions require you to connect polar form, conjugates, geometric reasoning, Vieta's formulas, and algebraic manipulation in sequence. The questions from the IB Problem are labeled [IBP] throughout.

IB TIP

Formula booklet entries for this section: Vieta's formulas (memorise), conjugate root theorem (memorise).

6.1 IB Problem Question 1 — Triangle Z_1OZ_2

The setup: Z_1 and Z_2 are points on an Argand diagram representing $z_1 = r_1e^{i\alpha}$ and $z_2 = r_2e^{i\theta}$. O is the origin. The triangle Z_1OZ_2 is described anticlockwise with $0 < \alpha, \theta < 2\pi$ and $0 < \alpha - \theta < \pi$. z_1 and z_2 are roots of $z^2 + az + b = 0$ where $a, b \in \mathbb{R}$.

Diagram — triangle Z_1OZ_2 on the Argand diagram

```
         Im
          |    Z₁ = r₁e^(iα)
          |   /↗ ← angle α from Re axis
          |  /
          | /  Z₂ = r₂e^(iθ)
          |/ ↗ ← angle θ from Re axis
  ────────O──────────────────────── Re
          |
          |  • Angles α and θ measured anticlockwise from posit
          |  • 0 < θ < α < 2π  (Z₁ is above Z₂ in the diagram)
          |  • α − θ = the angle at O inside the triangle
          |
  In parts (c)(d)(e): Z₁OZ₂ is an EQUILATERAL triangle (all sid
```

WORKED EXAMPLE

IB Problem — Q1a [2 marks]

Show that $z_1z_2^* = r_1r_2e^{i(\alpha-\theta)}$ where z_2^* is the complex conjugate of z_2.

Worked Solution — Product with conjugate in polar form

Step 1: $z_1 = r_1e^{i\alpha}$ and $z_2^* = r_2e^{-i\theta}$ [conjugate negates argument] — write in polar

Step 2: $z_1 \cdot z_2^* = r_1e^{i\alpha} \cdot r_2e^{-i\theta}$ — multiply

Step 3: $= r_1r_2 \cdot e^{i(\alpha-\theta)}$ ✓ — add exponents: $i\alpha + (-i\theta) = i(\alpha - \theta)$

WORKED EXAMPLE

IB Problem — Q1b [2 marks]

Given $\mathrm{Re}(z_1 z_2^*) = 0$, *show that* $Z_1 O Z_2$ *is a right-angled triangle*.

Worked Solution — $\mathrm{Re} = 0$ **means perpendicular**

Step 1: From Q1a: $z_1 z_2^* = r_1 r_2 \cdot e^{i(\alpha-\theta)}$

Step 2: $\mathrm{Re}(z_1 z_2^*) = r_1 r_2 \cdot \cos(\alpha - \theta)$ — real part of $re^{i\varphi}$ is $r\cos\varphi$

Step 3: $\mathrm{Re} = 0$ and $r_1, r_2 > 0 \to \cos(\alpha - \theta) = 0 \to \alpha - \theta = \frac{\pi}{2}$ — solve for angle

Step 4: The angle at O between OZ_1 and OZ_2 is $\frac{\pi}{2} = 90^\circ$ — geometric conclusion

Step 5: Therefore $Z_1 O Z_2$ has a right angle at O ✓

WORKED EXAMPLE

IB Problem — Q1c.i [2 marks]

For equilateral triangle $Z_1 O Z_2$*: express* z_1 *in terms of* z_2.

Worked Solution — Equilateral triangle condition

Step 1: Equilateral: all three sides equal, so $|z_1| = |z_2| \to r_1 = r_2$ — all sides equal

Step 2: Equilateral also means all angles $= 60^\circ$, so angle at $O = \frac{\pi}{3}$ — all angles 60 degrees

Step 3: $\alpha - \theta = \frac{\pi}{3} \to \alpha = \theta + \frac{\pi}{3}$ — angle between z_1 and z_2

Step 4: $z_1 = r_1 e^{i\alpha} = r_2 e^{i(\theta+\pi/3)} = r_2 e^{i\theta} \cdot e^{i\pi/3} = z_2 \cdot e^{i\pi/3}$ — factor

Step 5: $e^{i\pi/3} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + \frac{i\sqrt{3}}{2}$ — exact value

Step 6: $z_1 = z_2\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$ ✓ — final form

WORKED EXAMPLE

IB Problem — Q1c.ii [4 marks]

Hence show that $z_1^2 + z_2^2 = z_1 z_2$.

Worked Solution — Algebraic identity for equilateral triangle

Step 1: From c.i: $z_1 = z_2 \cdot e^{i\pi/3}$, so let $w = z_1/z_2 = e^{i\pi/3}$ — divide

Step 2: $z_1^2 + z_2^2 = z_1 z_2 \longleftrightarrow$ divide both sides by z_2^2:

$(z_1/z_2)^2 + 1 = (z_1/z_2) \to w^2 - w + 1 = 0$ — in terms of w

Step 3: Verify: $w = e^{i\pi/3}$, $w^2 = e^{2i\pi/3}$

$w^2 - w + 1 = e^{2i\pi/3} - e^{i\pi/3} + 1$

$= \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) - \left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) + 1 = \left(-\frac{1}{2} - \frac{1}{2} + 1\right) + i\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) = 0$ ✓ — compute

WORKED EXAMPLE

IB Problem — Q1d [5 marks]

z_1 and z_2 are roots of $z^2 + az + b = 0$ ($a, b \in \mathbb{R}$). Use result from (c)(ii) to show $a^2 - 3b = 0$.

Worked Solution — Vieta's formulas + equilateral condition

Step 1: Vieta's formulas for $z^2 + az + b = 0$:

$z_1 + z_2 = -a$ and $z_1 z_2 = b$ — sum and product of roots

Step 2: From c.ii: $z_1^2 + z_2^2 = z_1 z_2$ — use proven result

Step 3: $(z_1 + z_2)^2 - 2z_1 z_2 = z_1 z_2$ — expand $z_1^2 + z_2^2 = (z_1 + z_2)^2 - 2z_1 z_2$

Step 4: $(-a)^2 - 2b = b$ — substitute Vieta

Step 5: $a^2 - 2b = b \to a^2 = 3b \to a^2 - 3b = 0$ ✓ — rearrange

WORKED EXAMPLE

IB Problem — Q1e [3 marks]

$z^2 + az + 12 = 0$, $a \in \mathbb{R}$. Given $0 < \alpha - \theta < \pi$, deduce only one equilateral triangle Z_1OZ_2 can be formed.

Worked Solution — Uniqueness of equilateral triangle

Step 1: $b = 12$ (constant term of quadratic). From d: $a^2 = 3b = 36$, so $a = \pm 6$

Step 2: Both values give a valid quadratic, but check which gives $0 < \alpha - \theta < \pi$

Step 3: $a = -6$: $z^2 - 6z + 12 = 0 \to z = \frac{6 \pm \sqrt{36-48}}{2} = 3 \pm i\sqrt{3}$ — quadratic formula

$z_1 = 3 + i\sqrt{3}$: $r = 2\sqrt{3}, \alpha = \frac{\pi}{6}$. $z_2 = 3 - i\sqrt{3}$: $r = 2\sqrt{3}, \theta = -\frac{\pi}{6}$

$\alpha - \theta = \frac{\pi}{6} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{3}$. Is $0 < \frac{\pi}{3} < \pi$? YES ✓

Step 4: $a = 6$: $z^2 + 6z + 12 = 0 \to z = \frac{-6 \pm i\sqrt{12}}{2} = -3 \pm i\sqrt{3}$

$z_1 = -3 + i\sqrt{3}$: $\alpha = \frac{5\pi}{6}$. $z_2 = -3 - i\sqrt{3}$: $\theta = -\frac{5\pi}{6}$ (or $\frac{7\pi}{6}$)

$\alpha - \theta = \frac{5\pi}{6} - \left(-\frac{5\pi}{6}\right) = \frac{5\pi}{3}$. Is $0 < \frac{5\pi}{3} < \pi$? NO — exceeds π — fails constraint

Step 5: Only $a = -6$ satisfies $0 < \alpha - \theta < \pi$. **Exactly one equilateral triangle.** ✓

6.2 IB Problem Question 2 — Cube Roots of Unity (ω)

The setup: ω is a non-real solution of $z^3 = 1$, so $\omega^3 = 1$ and $\omega \neq 1$. This means ω satisfies $\omega^2 + \omega + 1 = 0$, giving the fundamental identity $1 + \omega + \omega^2 = 0$. Nearly every simplification in this question uses this identity. The second part involves $p = 1 - 3i$ and $q = x + (2x + 1)i$ with $x \in \mathbb{R}$.

Cube root of unity — everything needed for Q2

Identity	Notes
$\omega^3 = 1, \omega \neq 1$	ω satisfies $\omega^2 + \omega + 1 = 0$
$1 + \omega + \omega^2 = 0$	THE KEY IDENTITY (use this constantly)
$\omega + \omega^2 = -1$	direct consequence
$\omega \cdot \omega^2 = \omega^3 = 1$	product
$\omega^* = \omega^2$ and $(\omega^*)^2 = \omega$	they are complex conjugates
$\omega^2 = \omega^{-1}$	since $\omega^3 = 1 \to \omega \cdot \omega^2 = 1$
Powers cycle (period 3)	$\omega^3 = 1, \omega^4 = \omega, \omega^5 = \omega^2, \omega^6 = 1, \ldots$

WORKED EXAMPLE

IB Problem — Q2a(i) [4 marks]

Determine the value of $1 + \omega + \omega^2$.

Answer:

ω is a non-real root of $z^3 = 1$.

Factor: $z^3 - 1 = (z - 1)(z^2 + z + 1) = 0$

For $\omega \neq 1$: $\omega^2 + \omega + 1 = 0 \to 1 + \omega + \omega^2 = 0$

WORKED EXAMPLE

IB Problem — Q2a(ii) [4 marks]

Determine the value of $1 + \omega^* + (\omega^*)^2$.

Worked Solution — Using $\omega^* = \omega^2$

Step 1: $\omega^* = \omega^2$ (conjugate of a non-real cube root of unity is the other one) — key fact

Step 2: $1 + \omega^* + (\omega^*)^2 = 1 + \omega^2 + (\omega^2)^2 = 1 + \omega^2 + \omega^4$ — substitute

Step 3: $\omega^4 = \omega^3 \cdot \omega = 1 \cdot \omega = \omega$ — power cycle

Step 4: $= 1 + \omega^2 + \omega = 1 + \omega + \omega^2 = 0$ ✓ — apply main identity

WORKED EXAMPLE

IB Problem — Q2b [4 marks]

Show that $(\omega - 3\omega^2)(\omega^2 - 3\omega) = 13$.

Worked Solution — Expand and simplify using $1 + \omega + \omega^2 = 0$

Step 1: Expand: $(\omega - 3\omega^2)(\omega^2 - 3\omega)$

$= \omega \cdot \omega^2 - 3\omega \cdot \omega - 3\omega^2 \cdot \omega^2 + 9\omega^2 \cdot \omega$ — FOIL

$= \omega^3 - 3\omega^2 - 3\omega^4 + 9\omega^3$ — collect terms

Step 2: Substitute $\omega^3 = 1$ and $\omega^4 = \omega$:

$= 1 - 3\omega^2 - 3\omega + 9 \cdot 1$

$= 10 - 3(\omega + \omega^2)$ — factor

Step 3: Substitute $\omega + \omega^2 = -1$:

$= 10 - 3(-1) = 10 + 3 = 13$ ✓ — apply identity

WORKED EXAMPLE

IB Problem — Q2c [5 marks]

$p = 1 - 3i$, $q = x + (2x+1)i$, $x \in \mathbb{R}$. *Find all values of* x *such that* $|p| = |q|$.

Worked Solution — Equal moduli equation

Step 1: $|p|^2 = 1^2 + (-3)^2 = 1 + 9 = 10$ — compute $|p|$

Step 2: $|q|^2 = x^2 + (2x+1)^2 = x^2 + 4x^2 + 4x + 1 = 5x^2 + 4x + 1$ — expand $|q|$

Step 3: Set $|p|^2 = |q|^2$: $5x^2 + 4x + 1 = 10$ — equal moduli

Step 4: $5x^2 + 4x - 9 = 0$ — rearrange

Step 5: Discriminant: $16 + 4(5)(9) = 16 + 180 = 196 = 14^2$

Step 6: $x = \frac{-4 \pm 14}{10} \to x = \frac{10}{10} = 1$ or $x = \frac{-18}{10} = -\frac{9}{5}$ — quadratic formula

Step 7: Verify: $x = 1$: $|q|^2 = 5 + 4 + 1 = 10$ ✓. $x = -\frac{9}{5}$: $\frac{5 \cdot 81}{25} + \frac{4 \cdot (-9)}{5} + 1 = \frac{81}{5} - \frac{36}{5} + \frac{5}{5} = \frac{50}{5} = 10$ ✓

WORKED EXAMPLE

IB Problem — Q2d [6 marks]

Solve the inequality $\mathrm{Re}(pq) + 8 < (\mathrm{Im}(pq))^2$.

Worked Solution — Compute pq then solve inequality

Step 1: $p = 1 - 3i, q = x + (2x+1)i$ — given

Step 2: $pq = (1-3i)(x + (2x+1)i)$ — multiply out

$= x + (2x+1)i - 3xi - 3i(2x+1)i$ — expand

$= x + (2x+1)i - 3xi + 3(2x+1)$ [since $-3i \cdot (2x+1)i = -3(2x+1)i^2 = +3(2x+1)$] — $i^2 = -1$

$= [x + 6x + 3] + [(2x+1) - 3x]i$ — collect real and imag

$= (7x+3) + (1-x)i$ — final form

Step 3: $\mathrm{Re}(pq) = 7x+3, \mathrm{Im}(pq) = 1-x$ — read off parts

Step 4: Inequality: $(7x+3) + 8 < (1-x)^2$ — substitute

Step 5: $7x + 11 < 1 - 2x + x^2$ — expand right side

Step 6: $0 < x^2 - 9x - 10 \to 0 < (x-10)(x+1)$ — rearrange and factor

Step 7: Parabola $(x-10)(x+1) > 0$ when $x < -1$ or $x > 10$ — sign analysis

Step 8: Solution: $x < -1$ **or** $x > 10$

EXAM ALERT

For Q2d Step 7: sketch the parabola $y = (x-10)(x+1)$. It opens upward, crosses zero at $x = -1$ and $x = 10$. The expression is positive outside the roots — that is, for $x < -1$ or $x > 10$.

6.3 Key Proof Techniques — Quick Reference

Situation	Method
Show result is **REAL**	Compute it directly and show $\text{Im} = 0$. OR show $z = z^*$. OR show $z = \|w\|^2$ for some w.
Show result is **PURELY IMAGINARY**	Show $\text{Re}(z) = 0$, i.e. $z + z^* = 0$.
Show result is **ZERO**	Use $1 + \omega + \omega^2 = 0$ (cube roots) or sum of nth roots $= 0$.
Find when z^n is real	Write $z = r \cdot e^{i\theta}$. Then $z^n = r^n \cdot e^{in\theta}$. Real when $\sin(n\theta) = 0$, i.e. $n\theta = k\pi$.
Find when z^n is purely imaginary	Real part zero when $\cos(n\theta) = 0$, i.e. $n\theta = \frac{\pi}{2} + k\pi$.
Vieta's formulas for $z^2 + az + b = 0$	Sum of roots $= -a$. Product of roots $= b$.
Perpendicular lines on Argand diagram	$\text{Re}(z_1 z_2^*) = 0 \leftrightarrow$ angle between OZ_1 and OZ_2 is $\frac{\pi}{2}$.
Equilateral triangle	$z_1 = z_2 \cdot e^{\pm i\pi/3}$. Gives $z_1^2 + z_2^2 = z_1 z_2$ and $a^2 = 3b$.
Conjugate pairs	If $z = a + bi$ satisfies a polynomial with real coefficients, so does $z^* = a - bi$.
$\arg(z_1/z_2) = \arg(z_1) - \arg(z_2)$	Use this to find angles in triangles on the Argand diagram.

6.4 Master Formula Reference

Formula	Meaning / Key Point
$z = a + bi$	Cartesian form. $\text{Re}(z) = a, \text{Im}(z) = b$.
$z = r \,\text{cis}\, \theta = re^{i\theta}$	Polar/Euler form. $r = \|z\|, \theta = \arg(z)$.
$\|z\| = \sqrt{a^2 + b^2}$	Modulus — distance from origin.
$\arg(z)$ — quadrant-adjusted	Q1: $\arctan(b/a)$ Q2: $\pi - \arctan$ Q3: $-\pi + \arctan$ Q4: $-\arctan$
$z^* = a - bi = re^{-i\theta}$	Conjugate: flip imag part OR negate argument.
$z \cdot z^* = \|z\|^2$	Always real and ≥ 0.
$\|z_1 z_2\| = \|z_1\|\|z_2\|, \arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$	Polar multiplication.
$(r \,\text{cis}\, \theta)^n = r^n \,\text{cis}(n\theta)$	De Moivre: scale modulus^n, multiply argument by n .
$z + 1/z = 2\cos\theta$ ($\|z\|\| = 1$)	Technique for trig identities and integration.
$r^{1/n} \cdot \text{cis}\left(\frac{\theta + 2\pi k}{n}\right)$	nth root: $k = 0$ to $n - 1$. Equally spaced on circle of radius $r^{1/n}$.
Sum of nth roots of unity $= 0$	Vieta on $z^n - 1 = 0$. For cube: $1 + \omega + \omega^2 = 0$.
Vieta: $z^2 + az + b = 0$	Sum of roots $= -a$. Product $= b$.

6.5 Common Mistakes — Final Checklist

Mistake	Correct Approach
Wrong quadrant for $\arg(z)$	Always sketch the point. Q2 gives arg $> \pi/2$, Q3/Q4 give negative arg. Never blindly trust $\arctan(b/a)$.
Argument outside $(-\pi, \pi]$	After operations, check if arg is outside $(-\pi, \pi]$. If result is e.g. $7\pi/6$, rewrite as $-5\pi/6$.
Missing constraint on $\arg(z-a) = \theta$	This is a RAY, not a full line. State e.g. '$x > \text{Re}(a)$' to specify direction.
Integral middle term error	$\int \cos^5\theta \, d\theta$ middle term is $\frac{5\sin 3\theta}{48}$ NOT $\frac{\sin 3\theta}{16}$. The 5 from binomial stays.
Powers of i mistakes	$i^2 = -1, i^3 = -i, i^4 = 1$. In expansions, track EVERY power of i .
Vieta sign error	$z^2 + az + b = 0$: sum $= -a$ (MINUS a), product $= +b$.
Forgetting $\omega + \omega^2 = -1$	Almost every ω simplification requires substituting $\omega + \omega^2 = -1$ or $\omega^2 = -1 - \omega$.
Confusing locus types	For loci: $\|z - a\|\| = r$ is a CIRCLE (not a line). $\|z - a\|\| = \|z - b\|$ is a PERPENDICULAR BISECTOR.

IB Mathematics AA HL — Complex Numbers | Categories 1-6 | Review 14A, Oxford Key, IB Problem

Formula Booklet — Complex Numbers

This section lists all formulas from the IB Math AA HL formula booklet that are relevant to complex numbers. Formulas marked **GIVEN** are provided in the exam. Formulas marked **MEMORISE** must be known from memory.

Modulus and Argument

Formula	Status	Notes
$\|z\| = \sqrt{a^2 + b^2}$	**MEMORISE**	Not in formula booklet
$z = r(\cos\theta + i\sin\theta) = r\,\text{cis}\,\theta$	**GIVEN**	Polar form
$z = re^{i\theta}$	**GIVEN**	Euler form

Operations in Polar Form

Formula	Status	Notes
$z_1 z_2 = r_1 r_2\,\text{cis}(\theta_1 + \theta_2)$	**GIVEN**	Multiplication
$\frac{z_1}{z_2} = \frac{r_1}{r_2}\,\text{cis}(\theta_1 - \theta_2)$	**GIVEN**	Division

De Moivre's Theorem

Formula	Status	Notes
$(r\,\text{cis}\,\theta)^n = r^n\,\text{cis}(n\theta)$	**GIVEN**	Powers
nth roots: $w^{1/n} = r^{1/n}\,\text{cis}\left(\frac{\theta + 2\pi k}{n}\right)$	**GIVEN**	$k = 0, 1, \ldots, n-1$

Conjugate and Arithmetic

Formula	Status	Notes
$z^* = a - bi$	**MEMORISE**	Conjugate
$z \cdot z^* = \|z\|^2$	**MEMORISE**	Always real
$z + z^* = 2\text{Re}(z)$	**MEMORISE**	Always real
$z - z^* = 2i\,\text{Im}(z)$	**MEMORISE**	Always imaginary

Roots of Unity

Formula	Status	Notes
$1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0$	**MEMORISE**	Sum of nth roots of unity
$\omega^n = 1$	**MEMORISE**	Definition

Trig Identities via Complex Numbers

Formula	Status	Notes
$z + z^{-1} = 2\cos\theta$ (when $\|z\| = 1$)	**MEMORISE**	Key technique
$z - z^{-1} = 2i\sin\theta$ (when $\|z\| = 1$)	**MEMORISE**	Key technique

Loci

Formula	Status	Notes
$\|z - a\| = r \rightarrow$ Circle	**MEMORISE**	Centre a, radius r
$\|z - a\| = \|z - b\| \rightarrow$ Perpendicular bisector	**MEMORISE**	Of segment ab
$\arg(z - a) = \theta \rightarrow$ Ray	**MEMORISE**	From point a at angle θ

IB TIP

In the IB exam, you receive a formula booklet. Formulas marked GIVEN above are in it — you don't need to memorise them. But you DO need to know how and when to use them. Formulas marked MEMORISE are NOT in the booklet and must be learned.