IB Math AA HL Calculus — Topic 5 Notes & Worked Examples

IB HL Study Guide

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IB Math AA HL — Calculus

Complete Study Guide

Topics Covered

Topic 5 of the IB Math AA HL syllabus — Paper 2 and Paper 3

Videos on this page: Differentiation · Integration

▸Watch: Differentiation — Chain, Product, Quotient Rules & Applications VIDEO
▸Watch: Integration — Techniques, Areas, Volumes & Substitutions VIDEO

IB TIP

How to approach calculus on exams: The IB rewards structured method. Even if your final answer is wrong, showing correct working earns method marks. Always write the rule or theorem you are applying before you apply it, and show every intermediate line. A one-line answer — even if numerically correct — can score zero if no method is visible.

MEMORISE THIS

What is and is not in the formula booklet: Derivatives of $\sin x$, $\cos x$, e^x, $\ln x$, and x^n are given. The product rule, quotient rule, and chain rule are given. Standard integrals (including $\int \frac{1}{x}\,dx$, $\int e^x\,dx$, $\int \sin x\,dx$, $\int \cos x\,dx$) are given. The integration by parts formula is given. **NOT given:** L'Hôpital's rule, the formula for volumes of revolution, the Maclaurin series of e^x, $\sin x$, $\cos x$, and $\ln(1+x)$ (you must know how to derive them), and all kinematics interpretations.

Section 1: Limits and Continuity

A **limit** describes the value a function approaches as the input approaches some value, without necessarily reaching it. The notation $\lim_{x\to a} f(x) = L$ means "as x gets arbitrarily close to a (from either side), $f(x)$ gets arbitrarily close to L." Limits are the rigorous foundation for both derivatives (limit of a difference quotient) and integrals (limit of a Riemann sum).

IB TIP

Formula booklet entries for this section: L'Hôpital's rule is NOT in the booklet — you must state it explicitly when using it. The definition of the derivative as a limit IS given.

1.1 Evaluating Limits

Direct substitution works whenever f is continuous at a:

$\lim_{x \to a} f(x) = f(a) \qquad \text{(if } f \text{ is continuous at } a\text{)}$

When direct substitution gives $\frac{0}{0}$ or $\frac{\infty}{\infty}$ (an **indeterminate form**), use one of three strategies:

Strategy	When to use	Method
Factoring	Rational functions, $\frac{0}{0}$	Factor numerator and denominator, cancel common factor
Rationalising	Square roots, $\frac{0}{0}$	Multiply by conjugate over conjugate
L'Hôpital's rule	$\frac{0}{0}$ or $\frac{\infty}{\infty}$	Differentiate top and bottom separately, then substitute

One-sided limits: $\lim_{x \to a^-} f(x)$ (approach from the left) and $\lim_{x \to a^+} f(x)$ (from the right). The two-sided limit exists if and only if both one-sided limits exist and are equal.

Infinite limits and limits at infinity:

$$\lim_{x \to \infty} \frac{p(x)}{q(x)} = \begin{cases} \dfrac{\text{leading coeff of } p}{\text{leading coeff of } q} & \deg p = \deg q \\ 0 & \deg p < \deg q \\ \pm\infty & \deg p > \deg q \end{cases}$$

WORKED EXAMPLE

Limits by factoring

Evaluate $\lim\limits_{x \to 3} \dfrac{x^2 - 9}{x - 3}$.

Step 1: Direct substitution gives $\frac{9-9}{3-3} = \frac{0}{0}$ — indeterminate form.

Step 2: Factor the numerator: $x^2 - 9 = (x-3)(x+3)$.

Step 3: Cancel: $\dfrac{(x-3)(x+3)}{x-3} = x + 3$ for $x \neq 3$.

Step 4: Substitute: $\lim\limits_{x \to 3}(x+3) = 6$

$\lim_{x \to 3} \frac{x^2-9}{x-3} = 6$

1.2 L'Hôpital's Rule HL

Sometimes when you plug a value into a fraction, you get $\frac{0}{0}$ or $\frac{\infty}{\infty}$ — which tells you nothing about the limit. L'Hôpital's Rule gives you a way out: instead of the original fraction, take the derivative of the top and bottom separately, then try the limit again.

If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ or both equal $\pm\infty$, then:

$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

provided the right-hand limit exists. You may apply L'Hôpital's rule repeatedly until the indeterminate form resolves.

⚠ **EXAM ALERT**

L'Hôpital's rule applies **only** when the limit is in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. If the form is $\frac{3}{0}$, the limit is $\pm\infty$ (or does not exist) — L'Hôpital's rule does NOT apply. Always verify the form before using it.

WORKED EXAMPLE

L'Hôpital's Rule

Evaluate $\lim_{x \to 0} \dfrac{\sin x}{x}$.

Step 1: Direct substitution: $\frac{\sin 0}{0} = \frac{0}{0}$ — L'Hôpital applies.

Step 2: Differentiate numerator and denominator separately:

$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = \cos 0 = 1$

This fundamental limit appears inside proofs of derivative formulas, so knowing the result $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$ by heart saves time.

WORKED EXAMPLE

Repeated L'Hôpital

Evaluate $\lim_{x \to 0} \dfrac{e^x - 1 - x}{x^2}$.

Step 1: $\frac{e^0 - 1 - 0}{0} = \frac{0}{0}$ — apply L'Hôpital.

Step 2: $\lim_{x \to 0} \dfrac{e^x - 1}{2x}$ — still $\frac{0}{0}$, apply again.

Step 3: $\lim_{x \to 0} \dfrac{e^x}{2} = \dfrac{1}{2}$

1.3 Continuity

Continuity captures the idea that you can draw a function's graph without lifting your pen — no jumps, holes, or sudden breaks. It matters because many important theorems (like the Intermediate Value Theorem) only work when a function is continuous.

A function f is **continuous at** $x = a$ if all three conditions hold:

1. $f(a)$ is defined
2. $\lim_{x \to a} f(x)$ exists (both one-sided limits are equal)
3. $\lim_{x \to a} f(x) = f(a)$

Types of discontinuity:

Type	Description	Example
Removable	Limit exists, but $f(a)$ is missing or wrong	$f(x) = \frac{x^2-1}{x-1}$ at $x = 1$
Jump	One-sided limits exist but differ	Piecewise function with gap
Infinite	Limit is $\pm\infty$	$f(x) = \frac{1}{x}$ at $x = 0$

Intermediate Value Theorem (IVT): If f is continuous on $[a, b]$ and k is any value strictly between $f(a)$ and $f(b)$, then there exists at least one $c \in (a, b)$ with $f(c) = k$. This is used to prove roots exist.

⚠ **EXAM ALERT**

The IVT proves existence — it tells you a root exists, not where it is. To show a root of f exists on (a, b), show $f(a)$ and $f(b)$ have opposite signs and state that f is continuous on $[a, b]$.

Section 2: Differentiation

The **derivative** of f at x measures the instantaneous rate of change — equivalently, the gradient of the tangent to $y = f(x)$ at that point. It is defined as a limit:

$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

This is called **differentiation from first principles**.

2.1 First Principles

WORKED EXAMPLE

Derivative of $f(x) = x^2$ from first principles

Step 1: Write the difference quotient:

$\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^2-x^2}{h}$

Step 2: Expand: $= \dfrac{x^2+2xh+h^2-x^2}{h} = \dfrac{2xh+h^2}{h} = 2x+h$

Step 3: Take the limit: $f'(x) = \lim_{h\to 0}(2x+h) = 2x$

EXAM ALERT

In Paper 1 questions asking for first principles, you **must** write the limit definition with $\lim_{h\to 0}$, expand fully, cancel h from numerator and denominator, and then substitute $h = 0$. Skipping any step loses marks. The expression $\frac{f(x+h)-f(x)}{h}$ must appear explicitly.

2.2 Standard Derivatives

MEMORISE THIS

Standard Derivative Table

Function $f(x)$	Derivative $f'(x)$	Notes
x^n	nx^{n-1}	All real n, including fractions and negatives
e^x	e^x	Unique self-derivative
e^{kx}	ke^{kx}	Chain rule applied
a^x	$a^x \ln a$	$a > 0, a \neq 1$
$\ln x$	$\dfrac{1}{x}$	$x > 0$
$\ln\lvert x\rvert$	$\dfrac{1}{x}$	All $x \neq 0$
$\log_a x$	$\dfrac{1}{x \ln a}$	Change of base
$\sin x$	$\cos x$	Radians only
$\cos x$	$-\sin x$	Note the minus sign
$\tan x$	$\sec^2 x$	Memorise
$\sec x$	$\sec x \tan x$	Memorise
$\csc x$	$-\csc x \cot x$	Memorise
$\cot x$	$-\csc^2 x$	Memorise
$\arcsin x$	$\dfrac{1}{\sqrt{1-x^2}}$	$\lvert x\rvert < 1$
$\arccos x$	$-\dfrac{1}{\sqrt{1-x^2}}$	$\lvert x\rvert < 1$
$\arctan x$	$\dfrac{1}{1+x^2}$	All real x

⚠ **EXAM ALERT**

$\frac{d}{dx}(\cos x) = -\sin x$, NOT $+\sin x$. This sign error is one of the most common on IB Paper 1. Write it out explicitly every time until it is automatic.

2.3 The Three Combination Rules

Most functions you encounter are combinations of simpler pieces — two things multiplied, one divided by another, or one function plugged inside another. The three rules below tell you how to differentiate each of these combinations without expanding everything out first.

Power Rule (already covered above): $\frac{d}{dx}[x^n] = nx^{n-1}$

Product Rule: For $y = u(x) \cdot v(x)$:

$\frac{dy}{dx} = u'v + uv'$

Quotient Rule: For $y = \dfrac{u(x)}{v(x)}$:

$\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$

Chain Rule: For $y = f(g(x))$:

$\frac{dy}{dx} = f'(g(x)) \cdot g'(x) \qquad \text{or equivalently} \qquad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

⚠ **EXAM ALERT**

Quotient rule: the numerator is $u'v - uv'$ (top derivative first, then bottom derivative). The order is **not** interchangeable — $uv' - u'v$ gives the wrong sign. A memory aid: "low d-high minus high d-low, square the bottom and away you go."

WORKED EXAMPLE

Product Rule

Differentiate $y = x^3 e^{2x}$.

Let $u = x^3$, $v = e^{2x}$, so $u' = 3x^2$, $v' = 2e^{2x}$.

$\frac{dy}{dx} = 3x^2 \cdot e^{2x} + x^3 \cdot 2e^{2x} = e^{2x}(3x^2 + 2x^3) = x^2 e^{2x}(3 + 2x)$

Always factorise the final answer — IB mark schemes expect a simplified form.

WORKED EXAMPLE

Quotient Rule

Differentiate $y = \dfrac{\sin x}{x^2+1}$.

$u = \sin x, v = x^2+1, u' = \cos x, v' = 2x.$

$\frac{dy}{dx} = \frac{\cos x(x^2+1) - \sin x \cdot 2x}{(x^2+1)^2} = \frac{(x^2+1)\cos x - 2x\sin x}{(x^2+1)^2}$

WORKED EXAMPLE

Chain Rule

Differentiate $y = \sin(x^3+2x)$.

Let $u = x^3 + 2x$, so $y = \sin u$.

$\frac{dy}{dx} = \cos u \cdot \frac{du}{dx} = \cos(x^3+2x)\cdot(3x^2+2)$

WORKED EXAMPLE

Nested Chain Rule

Differentiate $y = e^{\sin(2x)}$.

Layer 1 (outermost): $\frac{d}{du}(e^u) = e^u$ where $u = \sin(2x)$

Layer 2: $\frac{d}{dv}(\sin v) = \cos v$ where $v = 2x$

Layer 3: $\frac{d}{dx}(2x) = 2$

$\frac{dy}{dx} = e^{\sin(2x)} \cdot \cos(2x) \cdot 2 = 2\cos(2x)\, e^{\sin(2x)}$

2.4 Implicit Differentiation

Sometimes a curve is defined by an equation like $x^2 + y^2 = 25$ where you can't easily isolate y — both variables are tangled together. Implicit differentiation lets you find the gradient anyway by differentiating both sides of the equation at once, without rearranging first.

When y is defined implicitly by an equation in x and y, differentiate both sides with respect to x, treating y as a function of x. Every time y is differentiated, multiply by $\dfrac{dy}{dx}$ (chain rule).

$\frac{d}{dx}[y^n] = ny^{n-1}\frac{dy}{dx} \qquad \frac{d}{dx}[\sin y] = \cos y \cdot \frac{dy}{dx}$

WORKED EXAMPLE

Implicit Differentiation

Find $\dfrac{dy}{dx}$ *for* $x^2 + y^2 = 25$.

Step 1: Differentiate both sides w.r.t. x:

$2x + 2y\frac{dy}{dx} = 0$

Step 2: Solve for $\frac{dy}{dx}$:

$\frac{dy}{dx} = -\frac{x}{y}$

This is the gradient at any point (x, y) on the circle. At $(3, 4)$: $\frac{dy}{dx} = -\frac{3}{4}$.

WORKED EXAMPLE

Implicit Differentiation — Product Term

Find $\dfrac{dy}{dx}$ *for* $x^2y + y^3 = 5$.

Differentiate each term:

- $\frac{d}{dx}(x^2y) = 2xy + x^2\frac{dy}{dx}$ (product rule)
- $\frac{d}{dx}(y^3) = 3y^2\frac{dy}{dx}$
- $\frac{d}{dx}(5) = 0$

Collect $\frac{dy}{dx}$ terms:

$2xy + x^2\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$

$\frac{dy}{dx}(x^2 + 3y^2) = -2xy$

$\frac{dy}{dx} = \frac{-2xy}{x^2+3y^2}$

EXAM ALERT

When differentiating a product like xy implicitly, the product rule gives TWO terms: $\frac{d}{dx}(xy) = y + x\frac{dy}{dx}$. Students frequently write just $x\frac{dy}{dx}$ and lose the y term.

2.5 Derivatives of Inverse Trigonometric Functions

Inverse trig functions like $\arctan x$ and $\arcsin x$ appear frequently in integration and in problems involving angles. Their derivatives look surprising at first, but each one can be derived using implicit differentiation — and they are worth knowing because they appear often as antiderivatives.

These arise from implicit differentiation of the definitions:

$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$ $\qquad$ $\frac{d}{dx}[\arctan x] = \frac{1}{1+x^2}$

With chain rule:

$\frac{d}{dx}[\arctan(g(x))] = \frac{g'(x)}{1+[g(x)]^2}$

Section 3: Applications of Differentiation

3.1 Tangent and Normal Lines

At the point $(a, f(a))$ on $y = f(x)$:

- **Tangent gradient:** $m_T = f'(a)$
- **Normal gradient:** $m_N = -\dfrac{1}{f'(a)}$ (negative reciprocal, since tangent $\perp$ normal)
- **Tangent equation:** $y - f(a) = f'(a)(x - a)$
- **Normal equation:** $y - f(a) = -\dfrac{1}{f'(a)}(x - a)$

⚠ EXAM ALERT

If $f'(a) = 0$ (horizontal tangent), the normal is a vertical line $x = a$ — it has no finite gradient. If $f'(a)$ is undefined (vertical tangent), the tangent is $x = a$ and the normal is horizontal with gradient 0. Never write the normal as $y = \frac{1}{0}x + c$.

WORKED EXAMPLE

Tangent and Normal

Find the equations of the tangent and normal to $y = x^3 - 2x$ at the point where $x = 2$.

Step 1: $y(2) = 8 - 4 = 4$ so the point is $(2, 4)$.

Step 2: $y' = 3x^2 - 2$, so $m_T = 3(4) - 2 = 10$.

Step 3: Tangent: $y - 4 = 10(x - 2) \Rightarrow y = 10x - 16$

Step 4: Normal: $m_N = -\frac{1}{10}$, so $y - 4 = -\frac{1}{10}(x - 2) \Rightarrow y = -\frac{x}{10} + \frac{21}{5}$

3.2 Stationary Points and Curve Sketching

A **stationary point** occurs where $f'(x) = 0$. The nature of a stationary point is determined by the **second derivative test** or by a **sign chart** on f':

Test	Local minimum	Local maximum	Inconclusive
Second derivative	$f''(a) > 0$	$f''(a) < 0$	$f''(a) = 0$
Sign chart on f'	f' changes $- \to +$	f' changes $+ \to -$	f' doesn't change sign

Increasing/decreasing: $f' > 0$ means increasing; $f' < 0$ means decreasing.

Concavity and inflection points:

Condition	Meaning
$f'' > 0$	Concave up (bowl shape)
$f'' < 0$	Concave down (cap shape)
f'' changes sign at $x = c$	Inflection point at $x = c$

⚠ EXAM ALERT

$f''(a) = 0$ does NOT mean $x = a$ is an inflection point. You must also verify that f'' **changes sign** at $x = a$. The function $f(x) = x^4$ has $f''(0) = 0$ but $x = 0$ is a minimum, not an inflection point.

Full curve-sketching checklist:

1. Domain and any restrictions
2. x-intercepts (set $y = 0$) and y-intercept (set $x = 0$)
3. Vertical asymptotes (values where denominator $= 0$)
4. Horizontal/oblique asymptotes (behaviour as $x \to \pm\infty$)
5. Stationary points: solve $f'(x) = 0$, classify each
6. Inflection points: solve $f''(x) = 0$, verify sign change in f''
7. Sketch, labelling all key features

WORKED EXAMPLE

Curve Sketching

Sketch $f(x) = \dfrac{x^2}{x^2 - 4}$*, identifying all key features.*

Domain: $x \neq \pm 2$

y-intercept: $f(0) = 0$

x-intercept: $x^2 = 0 \Rightarrow x = 0$

Vertical asymptotes: $x = 2$ and $x = -2$

Horizontal asymptote: As $x \to \pm\infty$, $f(x) \to 1$, so $y = 1$

Derivative: Using the quotient rule:

$f'(x) = \frac{2x(x^2-4) - x^2 \cdot 2x}{(x^2-4)^2} = \frac{-8x}{(x^2-4)^2}$

Stationary point at $x = 0$: $f(0) = 0$, $f''(0) > 0$ (local minimum at $(0, 0)$)

Behaviour: $f(x) > 1$ for $|x| > 2$; $f(x) \leq 0$ for $|x| < 2$.

3.3 Optimisation

Optimisation is how you use calculus to answer "what is the best possible outcome?" questions — the largest area you can enclose with a fixed fence, the box with maximum volume from a sheet of card, the speed that minimises fuel use. The key insight is that at a maximum or minimum, the rate of change equals zero, so you find it by solving $f'(x) = 0$.

Optimisation problems ask for the maximum or minimum value of some quantity. The standard approach:

1. Define the variable(s) and write the **objective function** (the quantity to optimise)
2. Use any constraint to write the objective function in terms of a single variable
3. Differentiate and solve $f'(x) = 0$
4. Use the second derivative or domain analysis to confirm it is a maximum/minimum
5. Answer the question — often the maximum/minimum value is required, not just x

⚠ **EXAM ALERT**

Always check the **domain** for the optimisation problem. A critical point at $x = 5$ is useless if the physical constraint restricts x to $[0, 4]$. In that case, check the endpoints as well.

WORKED EXAMPLE

Optimisation — Closed Box

A closed rectangular box has a square base of side x cm and height h cm. Its surface area is 600 cm^2. Find the dimensions that maximise the volume.

Objective function: $V = x^2 h$

Constraint: $2x^2 + 4xh = 600 \Rightarrow h = \dfrac{600 - 2x^2}{4x} = \dfrac{300 - x^2}{2x}$

Single-variable form:

$V(x) = x^2 \cdot \frac{300-x^2}{2x} = \frac{x(300-x^2)}{2} = 150x - \frac{x^3}{2}$

Differentiate: $V'(x) = 150 - \frac{3x^2}{2}$

Set to zero: $150 = \frac{3x^2}{2} \Rightarrow x^2 = 100 \Rightarrow x = 10$ (positive dimension)

Confirm maximum: $V''(x) = -3x < 0$ for $x > 0$ — concave down, so this is a maximum.

Dimensions: $x = 10$ cm, $h = \frac{300-100}{20} = 10$ cm. The optimal box is a cube.

$V_{\text{max}} = 1000 \text{ cm}^3$

3.4 Related Rates

When two quantities are connected by a formula, changing one forces the other to change too — and you can find exactly how fast. For example, if a balloon's radius is growing, how fast is its volume growing? Related rates use the chain rule to link these speeds together.

In related-rates problems, two or more quantities change with time. The chain rule links their rates of change.

Strategy:

1. Identify what is changing, assign variables and their rates (typically as $\frac{dy}{dt}$, $\frac{dx}{dt}$, etc.)
2. Write a geometric or physical relationship between the variables
3. Differentiate implicitly with respect to t
4. Substitute known values and solve

WORKED EXAMPLE

Related Rates — Expanding Circle

The radius of a circle is increasing at $3\ \text{cm s}^{-1}$. *Find the rate of increase of the area when the radius is* 5 *cm.*

Let r = radius, $A = \pi r^2$.

$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$

When $r = 5$ and $\frac{dr}{dt} = 3$:

$\frac{dA}{dt} = 2\pi(5)(3) = 30\pi \approx 94.2\ \text{cm}^2\text{s}^{-1}$

EXAM ALERT

Related rates questions require **units in the answer**. If r is in cm and t is in seconds, then $\frac{dA}{dt}$ is in cm^2s^{-1}. Missing units on a rates answer will lose the final mark.

3.5 Kinematics

For a particle moving in a straight line with displacement $s(t)$:

$v(t) = s'(t) = \frac{ds}{dt} \qquad a(t) = v'(t) = s''(t) = \frac{d^2s}{dt^2}$

Key interpretations:

Quantity	Sign	Meaning
$v > 0$	Positive	Moving in positive direction
$v < 0$	Negative	Moving in negative direction
$v = 0$	Zero	Particle is at rest (momentarily)
$a > 0$	Positive	Velocity increasing
$a < 0$	Negative	Velocity decreasing (decelerating if $v > 0$)

Speed is $|v|$. A particle **decelerates** when v and a have opposite signs.

Total distance (not displacement): integrate $|v(t)|$, or split the integral at points where $v = 0$ and add absolute values.

⚠ EXAM ALERT

Distance ≠ displacement. Displacement is $\int_a^b v(t)\,dt$ (signed). Distance is $\int_a^b |v(t)|\,dt$ (unsigned). If the particle reverses, the displacement integral underestimates the total path length. Always find where $v = 0$ to check for reversal.

WORKED EXAMPLE

Kinematics

A particle has displacement $s(t) = t^3 - 6t^2 + 9t$ *metres,* $t \geq 0$ *seconds. Find: (a) when it is at rest, (b) its acceleration when* $t = 3$*, (c) the total distance in* $0 \leq t \leq 4$*.*

Part (a): $v = 3t^2 - 12t + 9 = 3(t-1)(t-3) = 0 \Rightarrow t = 1$ or $t = 3$

Part (b): $a = v' = 6t - 12$. At $t = 3$: $a = 18 - 12 = 6 \text{ m s}^{-2}$

Part (c): Position values: $s(0) = 0$, $s(1) = 4$, $s(3) = 0$, $s(4) = 4$

Distances: $|4 - 0| + |0 - 4| + |4 - 0| = 4 + 4 + 4 = 12$ m

Section 4: Integration

Integration is the reverse of differentiation. The **indefinite integral** $\int f(x)\,dx$ gives a family of antiderivatives $F(x) + C$. The **definite integral** $\int_a^b f(x)\,dx = F(b) - F(a)$ gives a specific number (the net signed area under the curve).

Fundamental Theorem of Calculus:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x) \qquad \text{and} \qquad \int_a^b f'(x)\,dx = f(b) - f(a)$$

4.1 Standard Integrals

MEMORISE THIS

Standard Integral Table

Integrand $f(x)$	$\int f(x)\,dx$	Condition
x^n	$\frac{x^{n+1}}{n+1} + C$	$n \neq -1$
$\frac{1}{x}$	$\ln\|x\| + C$	$x \neq 0$
e^x	$e^x + C$	
e^{kx}	$\frac{1}{k}e^{kx} + C$	$k \neq 0$
$\sin x$	$-\cos x + C$	
$\cos x$	$\sin x + C$	
$\tan x$	$\ln\|\sec x\| + C$	
$\sec^2 x$	$\tan x + C$	
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x + C$	$\|x\| < 1$
$\frac{1}{1+x^2}$	$\arctan x + C$	
$\frac{1}{x^2+a^2}$	$\frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$	
$\frac{1}{\sqrt{a^2-x^2}}$	$\arcsin\left(\frac{x}{a}\right) + C$	

⚠ **EXAM ALERT**

$\int \frac{1}{x}\,dx = \ln|x| + C$, NOT $\ln(x) + C$. The absolute value is essential when x can be negative. On IB exams, omitting the absolute value bars inside a logarithm loses a mark.

4.2 Integration by Substitution

Substitution (reverse chain rule): choose $u = g(x)$, then $du = g'(x)\,dx$.

$\int f(g(x)) \cdot g'(x)\,dx = \int f(u)\,du$

Key step: Always express dx in terms of du by computing $\frac{du}{dx}$, then substitute. For definite integrals, either change the limits (using $u = g(x)$) or convert back to x at the end.

✎ **WORKED EXAMPLE**

Substitution — Indefinite

Find $\displaystyle\int 2x\cos(x^2)\,dx$.

Let $u = x^2$, so $du = 2x\,dx$.

$\int 2x\cos(x^2)\,dx = \int \cos u\,du = \sin u + C = \sin(x^2) + C$

WORKED EXAMPLE

Substitution — Definite

Evaluate $\int_0^1 \frac{x}{(x^2+1)^3}\,dx$.

Let $u = x^2 + 1$, $du = 2x\,dx \Rightarrow x\,dx = \frac{1}{2}du$.

Change limits: $x = 0 \Rightarrow u = 1$; $x = 1 \Rightarrow u = 2$.

$\int_1^2 \frac{1}{u^3} \cdot \frac{1}{2}\,du = \frac{1}{2}\int_1^2 u^{-3}\,du = \frac{1}{2}\left[\frac{u^{-2}}{-2}\right]_1^2 = -\frac{1}{4}\left[\frac{1}{4} - 1\right] = -\frac{1}{4}\cdot\left(-\frac{3}{4}\right) = \frac{3}{16}$

4.3 Integration by Parts

When you have an integral that is a product of two different types of functions — like $x \cdot e^x$ or $x \cdot \ln x$ — substitution doesn't work. Integration by parts is the technique for these cases: it splits the integral into a simpler one by trading complexity between the two factors.

The integration-by-parts formula follows from reversing the product rule:

$\int u\,dv = uv - \int v\,du$

Choosing u — LIATE order (highest priority to lowest):

Priority	Type	Example
1st	Logarithms	$\ln x$
2nd	Inverse trig	$\arctan x$
3rd	Algebraic	x^n
4th	Trigonometric	$\sin x$, $\cos x$
5th	Exponential	e^x

Make u the term with the higher-priority type; let dv be the rest.

EXAM ALERT

LIATE is a guide, not a law. For $\int xe^x\,dx$, take $u = x$ (algebraic, priority 3) and $dv = e^x\,dx$ (exponential, priority 5) — then $du = dx$ and $v = e^x$. If you take $u = e^x$ and $dv = x\,dx$, the integral becomes more complex, not simpler.

WORKED EXAMPLE

Integration by Parts — Standard

Find $\displaystyle\int xe^x\,dx$.

$u = x, dv = e^x\,dx \Rightarrow du = dx, v = e^x$

$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C = e^x(x-1) + C$

WORKED EXAMPLE

Integration by Parts — Twice (IBP^2)

Find $\displaystyle\int x^2e^x\,dx$.

First IBP: $u = x^2, dv = e^x\,dx \Rightarrow du = 2x\,dx, v = e^x$

$\int x^2e^x\,dx = x^2e^x - 2\int xe^x\,dx$

Second IBP: From above, $\int xe^x\,dx = e^x(x-1) + C$

$= x^2e^x - 2e^x(x-1) + C = e^x(x^2 - 2x + 2) + C$

WORKED EXAMPLE

Integration by Parts — Cyclic

Find $\displaystyle\int e^x\cos x\,dx$.

First IBP: $u = e^x, dv = \cos x\,dx \Rightarrow v = \sin x$

$I = e^x\sin x - \int e^x\sin x\,dx$

Second IBP: $u = e^x, dv = \sin x\,dx \Rightarrow v = -\cos x$

$I = e^x\sin x - \left(-e^x\cos x + \int e^x\cos x\,dx\right) = e^x\sin x + e^x\cos x - I$

Solve for I: $2I = e^x(\sin x + \cos x) \Rightarrow I = \dfrac{e^x(\sin x + \cos x)}{2} + C$

Practice: Integration by Parts — Fading Sequence

The worked examples above showed the full method. Now try these progressively: the first shows all steps, the second hides the final steps for you to attempt, and the third gives only the setup.

WORKED EXAMPLE *Full worked example — all steps shown*

Find $\displaystyle\int x\sin x\,dx$.

Step 1

Identify u and dv: Using LIATE, $u = x$ (algebraic) and $dv = \sin x\, dx$.

Step 2

Compute du and v: $du = dx, v = \int \sin x\, dx = -\cos x$

Step 3

Apply the formula: $\int x \sin x\, dx = x(-\cos x) - \int(-\cos x)\, dx = -x\cos x + \int \cos x\, dx$

Step 4

Evaluate the remaining integral: $= -x\cos x + \sin x + C$

YOUR TURN (PARTIAL) *Partial example — try the last steps yourself*

Find $\displaystyle\int x^2 \ln x\, dx$**.**

Steps 1—2 are shown. Try steps 3—4 before revealing.

Step 1

Identify u and dv: Using LIATE, $u = \ln x$ (log, highest priority) and $dv = x^2\, dx$.

Step 2

Compute du and v: $du = \frac{1}{x}\, dx, v = \frac{x^3}{3}$

Try it yourself, then click to reveal — Step 3 — Apply the IBP formula and simplify the remaining integral

Try it yourself, then click to reveal — Step 4 — Evaluate and write the final answer

YOUR TURN (SCAFFOLDED) *Scaffolded — only the setup is given*

Find $\displaystyle\int \ln x\, dx$**.**

Work through the full solution, then reveal each step to check.

Try it yourself, then click to reveal — Step 1 — Choose u and dv (hint: write ln x as ln x times 1)

Try it yourself, then click to reveal — Step 2 — Apply the IBP formula

Try it yourself, then click to reveal — Step 3 — Write the final answer

4.4 Partial Fractions HL

A fraction like $\frac{3x+1}{(x+1)(x-2)}$ is hard to integrate directly, but if you can split it into two simpler fractions each with a single linear denominator, each piece integrates to a natural log. Partial fractions is the technique that does this splitting.

Partial fractions decompose a rational function into simpler fractions before integrating. The method depends on the nature of the denominator's factors.

Case 1 — Distinct linear factors: $\dfrac{f(x)}{(x-a)(x-b)} = \dfrac{A}{x-a} + \dfrac{B}{x-b}$

Case 2 — Repeated linear factor: $\dfrac{f(x)}{(x-a)^2} = \dfrac{A}{x-a} + \dfrac{B}{(x-a)^2}$

Case 3 — Irreducible quadratic factor: $\dfrac{f(x)}{(x-a)(x^2+bx+c)} = \dfrac{A}{x-a} + \dfrac{Bx+C}{x^2+bx+c}$

Important: If the degree of the numerator $\geq$ degree of the denominator, perform polynomial long division first to obtain a proper fraction.

WORKED EXAMPLE

Partial Fractions Integration

Find $\displaystyle\int \frac{3x+1}{(x+1)(x-2)}\,dx$.

Step 1: Decompose: $\dfrac{3x+1}{(x+1)(x-2)} = \dfrac{A}{x+1} + \dfrac{B}{x-2}$

Step 2: Multiply through: $3x+1 = A(x-2) + B(x+1)$

Step 3: Substitute $x = 2$: $7 = 3B \Rightarrow B = \frac{7}{3}$

Step 4: Substitute $x = -1$: $-2 = -3A \Rightarrow A = \frac{2}{3}$

Step 5: Integrate:

$\int \frac{3x+1}{(x+1)(x-2)}\,dx = \frac{2}{3}\ln|x+1| + \frac{7}{3}\ln|x-2| + C$

EXAM ALERT

Before partial fractions, always check whether the rational function is **proper** (degree of numerator < degree of denominator). If it is improper, divide first. Attempting partial fractions on an improper fraction without dividing first gives a wrong decomposition.

Section 5: Applications of Integration

5.1 Area Under a Curve

The area between $y = f(x)$ and the x-axis from $x = a$ to $x = b$ is:

$A = \int_a^b |f(x)|\,dx$

If $f(x) \geq 0$ on $[a, b]$, this simplifies to $\int_a^b f(x)\,dx$. If f changes sign, split the interval at the zeros and add absolute values.

⚠ EXAM ALERT

$\int_a^b f(x)\,dx$ gives the **net signed area** (regions below the axis subtract). For a **total area**, split at roots and sum the absolute values. IB questions usually ask explicitly for "area" (unsigned) or "value of the integral" (signed) — do not confuse them.

5.2 Area Between Two Curves

For $f(x) \geq g(x)$ on $[a, b]$:

$$A = \int_a^b [f(x) - g(x)]\,dx$$

If the curves cross within $[a, b]$, find the crossing points (intersections), then split and add.

WORKED EXAMPLE

Area Between Two Curves

Find the area enclosed by $y = x^2$ *and* $y = x + 2$.

Step 1: Find intersections: $x^2 = x + 2 \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x-2)(x+1) = 0 \Rightarrow x = -1, 2$

Step 2: On $[-1, 2]$: $x + 2 \geq x^2$ (check at $x = 0$: $2 > 0$). So $f = x + 2, g = x^2$.

Step 3:

$$A = \int_{-1}^2 (x + 2 - x^2)\,dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^2$$

$$= \left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right) = \frac{10}{3} - \left(-\frac{7}{6}\right) = \frac{10}{3} + \frac{7}{6} = \frac{27}{6} = \frac{9}{2}$$

5.3 Volume of Revolution HL

If you take a curve and spin it around an axis, you carve out a 3D solid — like a vase or a bullet shape. The volume of revolution formula calculates that solid's volume by treating it as a stack of thin circular discs, each with radius equal to the function's value at that point.

Rotation about the x-axis: The volume generated by rotating $y = f(x)$ about the x-axis from $x = a$ to $x = b$:

$$V = \pi \int_a^b [f(x)]^2\,dx$$

Rotation about the y-axis: Express x as a function of y:

$$V = \pi \int_c^d [g(y)]^2\,dy$$

Volume between two curves (shell method or washer method):

$$V = \pi \int_a^b \left([f(x)]^2 - [g(x)]^2\right) dx \qquad (\text{washer method, outer}^2 - \text{inner}^2)$$

⚠ **EXAM ALERT**

The volume formula is $V = \pi \int y^2 \, dx$ — it is y^2, not y. Students commonly write $V = \pi \int y \, dx$. The factor of π comes from the cross-sectional area $\pi r^2 = \pi y^2$. This formula is NOT in the IB formula booklet for AA HL — you must recall it.

✎ **WORKED EXAMPLE**

Volume of Revolution

Find the volume generated when $y = \sqrt{x}$, $0 \le x \le 4$, *is rotated* 360° *about the* x*-axis.*

$$V = \pi \int_0^4 (\sqrt{x})^2 \, dx = \pi \int_0^4 x \, dx = \pi \left[\frac{x^2}{2}\right]_0^4 = \pi \cdot 8 = 8\pi$$

5.4 Kinematics — Integration

Given acceleration $a(t)$ or velocity $v(t)$:

$$v(t) = \int a(t) \, dt + C_1 \qquad s(t) = \int v(t) \, dt + C_2$$

The constants are found from **initial conditions**: $v(0) = v_0$ and $s(0) = s_0$.

Displacement over $[t_1, t_2]$**:** $\displaystyle\int_{t_1}^{t_2} v(t) \, dt = s(t_2) - s(t_1)$

Total distance over $[t_1, t_2]$**:** $\displaystyle\int_{t_1}^{t_2} |v(t)| \, dt$

✎ **WORKED EXAMPLE**

Kinematics — Finding Position from Acceleration

A particle starts at rest at the origin. Its acceleration is $a(t) = 6t - 4$. *Find its position at* $t = 3$.

Step 1: $v(t) = \int (6t - 4) \, dt = 3t^2 - 4t + C_1$. Initial rest: $v(0) = 0 \Rightarrow C_1 = 0$.

Step 2: $s(t) = \int (3t^2 - 4t) \, dt = t^3 - 2t^2 + C_2$. Starts at origin: $s(0) = 0 \Rightarrow C_2 = 0$.

Step 3: $s(3) = 27 - 18 = 9$ m

Section 6: Differential Equations

A differential equation is an equation that contains a function and its derivative at the same time — it describes how something changes rather than what it equals. They are the language of real-world modelling: population growth, radioactive decay, temperature cooling, and spread of disease are all described by differential equations.

A **differential equation (DE)** relates a function to its derivatives. The **order** is the highest derivative appearing. The **general solution** contains arbitrary constants; an **initial condition** (or **boundary condition**) pins down a **particular solution**.

6.1 Separable Differential Equations

The simplest differential equations to solve are ones where you can get all the y terms on one side and all the x terms on the other — then integrate both sides separately. This is called "separating variables."

A first-order DE is **separable** if it can be written as $\frac{dy}{dx} = f(x)g(y)$. Separate variables and integrate both sides:

$\int \frac{1}{g(y)}\, dy = \int f(x)\, dx$

WORKED EXAMPLE

Separable DE

Solve $\frac{dy}{dx} = xy$, *given* $y(0) = 2$.

Step 1: Separate: $\frac{dy}{y} = x\, dx$

Step 2: Integrate: $\ln|y| = \frac{x^2}{2} + C$

Step 3: Exponentiate: $|y| = e^C \cdot e^{x^2/2}$, so $y = Ae^{x^2/2}$ where $A = \pm e^C$

Step 4: Apply $y(0) = 2$: $2 = Ae^0 = A$

Particular solution: $y = 2e^{x^2/2}$

EXAM ALERT

After separating and integrating, you get $\ln|y| = \ldots + C$. Exponentiating gives $|y| = e^C e^{\cdots}$, and the $\pm$ from the absolute value is absorbed into a new constant A, so $y = Ae^{\cdots}$ where $A \neq 0$. Do not lose the $\pm$ or the constant — they combine to give A which is determined by the initial condition.

6.2 Initial Value Problems

A differential equation's general solution contains an unknown constant, giving a whole family of curves. An initial value problem pins down which specific curve you want by giving one known point on it — for example, the population at time zero, or the temperature at the start of an experiment.

An **initial value problem (IVP)** specifies the DE and a condition such as $y(x_0) = y_0$. The general solution's constant is uniquely determined by substituting x_0 and y_0.

WORKED EXAMPLE

Initial Value Problem

Solve $\frac{dy}{dx} = \frac{x}{y}$, $y(0) = 3$.

Separate: $y\,dy = x\,dx$

Integrate: $\frac{y^2}{2} = \frac{x^2}{2} + C$

Apply IC: $\frac{9}{2} = 0 + C \Rightarrow C = \frac{9}{2}$

Particular solution: $y^2 = x^2 + 9$, or $y = \sqrt{x^2 + 9}$ (positive since $y(0) = 3 > 0$)

6.3 Modelling with Differential Equations

Many natural phenomena are modelled by DEs. Key models:

Model	DE	Solution
Exponential growth	$\frac{dN}{dt} = kN, k > 0$	$N = N_0 e^{kt}$
Exponential decay	$\frac{dN}{dt} = -kN, k > 0$	$N = N_0 e^{-kt}$
Newton's Law of Cooling	$\frac{dT}{dt} = -k(T - T_{\text{env}})$	$T = T_{\text{env}} + (T_0 - T_{\text{env}})e^{-kt}$
Logistic growth	$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$	S-shaped curve, $P \to K$

EXAM ALERT

For Newton's Law of Cooling, let $\theta = T - T_{\text{env}}$ (excess temperature). Then $\frac{d\theta}{dt} = -k\theta$, which is a standard exponential decay. Always substitute the environmental temperature first before integrating.

WORKED EXAMPLE

Modelling — Radioactive Decay

A radioactive substance has half-life 8 years. Find: (a) the decay constant k, (b) the fraction remaining after 20 years.

Part (a): At $t = 8$, $N = \frac{1}{2}N_0$:

$$\frac{N_0}{2} = N_0 e^{-8k} \Rightarrow \frac{1}{2} = e^{-8k} \Rightarrow -8k = \ln\frac{1}{2} = -\ln 2 \Rightarrow k = \frac{\ln 2}{8}$$

Part (b):

$$\frac{N}{N_0} = e^{-20k} = e^{-20\ln 2/8} = e^{-5\ln 2/2} = 2^{-5/2} = \frac{1}{4\sqrt{2}} \approx 0.177$$

About 17.7% remains after 20 years.

Section 7: Maclaurin Series HL

Polynomials are easy to work with — you can add, multiply, differentiate, and integrate them by hand. A Maclaurin series takes a complicated function like e^x or $\sin x$ and rewrites it as an infinite polynomial. This makes it possible to approximate function values, evaluate limits that look like $\frac{0}{0}$, and integrate functions that have no closed-form antiderivative.

A **Maclaurin series** expresses a function as an infinite power series centred at $x = 0$:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

The series converges to $f(x)$ within a **radius of convergence** R. Outside this radius the series diverges.

7.1 Deriving the Four Standard Series

MEMORISE THIS

The Four Essential Maclaurin Series (must be able to derive)

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{(all } x\text{)}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad \text{(all } x\text{)}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad \text{(all } x\text{)}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} \qquad (-1 < x \le 1)$$

WORKED EXAMPLE

Deriving the Maclaurin Series for e^x

Let $f(x) = e^x$. Since $\frac{d}{dx}(e^x) = e^x$, all derivatives equal e^x.

At $x = 0$: $f^{(n)}(0) = e^0 = 1$ for all n.

Substituting into the Maclaurin formula:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

WORKED EXAMPLE

Deriving the Maclaurin Series for $\sin x$

$f(x) = \sin x$. Compute derivatives at $x = 0$:

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sin x$	0
1	$\cos x$	1
2	$-\sin x$	0
3	$-\cos x$	-1
4	$\sin x$	0

The pattern is $0, 1, 0, -1, 0, 1, \ldots$ Only odd powers survive:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

7.2 Applications of Maclaurin Series

Substitution: Replace x with a multiple or power to get new series.

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots$$

Evaluating limits: Series give an exact expansion near $x = 0$, resolving indeterminate forms.

Approximation: The first few terms give accurate approximations for small x.

Multiplying series: Multiply term by term, collecting powers.

WORKED EXAMPLE

Limit Using Maclaurin Series

Evaluate $\lim_{x \to 0} \dfrac{e^x - 1 - x - \frac{x^2}{2}}{x^3}$.

Substitute the series for e^x:

$e^x - 1 - x - \frac{x^2}{2} = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right) - 1 - x - \frac{x^2}{2} = \frac{x^3}{6} + \cdots$

$\lim_{x \to 0} \frac{x^3/6 + \cdots}{x^3} = \frac{1}{6}$

WORKED EXAMPLE

Integration Using Maclaurin Series

Find $\displaystyle\int_0^{0.1} e^{-x^2}\, dx$ *to 5 decimal places.*

Using $e^{-x^2} = 1 - x^2 + \dfrac{x^4}{2!} - \cdots$:

$\int_0^{0.1} e^{-x^2}\, dx \approx \int_0^{0.1} \left(1 - x^2 + \frac{x^4}{2}\right) dx = \left[x - \frac{x^3}{3} + \frac{x^5}{10}\right]_0^{0.1}$

$= 0.1 - \frac{0.001}{3} + \frac{0.00001}{10} \approx 0.1 - 0.000333 + 0.000001 = 0.099668$

EXAM ALERT

Maclaurin series for $\ln(1 + x)$ converges only for $-1 < x \leq 1$. At $x = 1$: the alternating harmonic series converges conditionally to $\ln 2$. At $x = -1$: it diverges. Always state the interval of validity when writing the series. Applying the series outside its radius gives nonsensical results.

WORKED EXAMPLE

Maclaurin Series Multiplication

Find the Maclaurin series for $\sin x \cdot e^x$ *up to and including the* x^3 *term.*

$\sin x \approx x - \frac{x^3}{6} + \cdots \qquad e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$

Multiply term by term, collecting up to x^3:

$x^1 : \ x \cdot 1 = x \ x^2 : \ x \cdot x = x^2 \ x^3 : \ x \cdot \frac{x^2}{2} + \left(-\frac{x^3}{6}\right) \cdot 1 = \frac{x^3}{2} - \frac{x^3}{6} = \frac{x^3}{3}$

$\sin x \cdot e^x \approx x + x^2 + \frac{x^3}{3} + \cdots$

Section 8: MCQ Practice

WORKED EXAMPLE

Q1. The derivative of $f(x) = \ln(\sin x)$ is:

(A) $\cos x$ **(B)** $\cot x$ **(C)** $\tan x$ **(D)** $\dfrac{1}{\sin x}$

Answer: (B) Chain rule: $f'(x) = \dfrac{1}{\sin x} \cdot \cos x = \dfrac{\cos x}{\sin x} = \cot x$.

WORKED EXAMPLE

Q2. $\displaystyle\int_0^{\pi/2} \sin^2 x \, dx =$

(A) 0 **(B)** $\dfrac{\pi}{4}$ **(C)** $\dfrac{\pi}{2}$ **(D)** 1

Answer: (B) Use $\sin^2 x = \dfrac{1 - \cos 2x}{2}$:

$\int_0^{\pi/2} \frac{1-\cos 2x}{2} \, dx = \left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{\pi/2} = \frac{\pi}{4} - 0 = \frac{\pi}{4}$

WORKED EXAMPLE

Q3. If $y = x^x$, then $\dfrac{dy}{dx} =$

(A) $x \cdot x^{x-1}$ **(B)** $x^x \ln x$ **(C)** $x^x(1 + \ln x)$ **(D)** $x^x \cdot x$

Answer: (C) Use logarithmic differentiation: $\ln y = x \ln x$.

$\frac{1}{y}\frac{dy}{dx} = \ln x + x \cdot \frac{1}{x} = \ln x + 1$

$\frac{dy}{dx} = y(\ln x + 1) = x^x(1 + \ln x)$

WORKED EXAMPLE

Q4. The value of $\displaystyle\lim_{x\to 0} \frac{1 - \cos x}{x^2}$ is:

(A) 0 **(B)** $\dfrac{1}{4}$ **(C)** $\dfrac{1}{2}$ **(D)** 1

Answer: (C) Using the Maclaurin series $\cos x = 1 - \dfrac{x^2}{2} + \cdots$:

$\frac{1-\cos x}{x^2} = \frac{x^2/2 - \cdots}{x^2} \to \frac{1}{2}$

Alternatively, apply L'Hôpital's rule twice: $\dfrac{\sin x}{2x} \to \dfrac{\cos x}{2} \to \dfrac{1}{2}$.

WORKED EXAMPLE

Q5. The function $f(x) = x^3 - 6x^2 + 9x$ has an inflection point at:

(A) $x = 1$ **(B)** $x = 2$ **(C)** $x = 3$ **(D)** $x = 0$

Answer: (B) $f'(x) = 3x^2 - 12x + 9$, $f''(x) = 6x - 12 = 6(x-2)$.

$f''(x) = 0$ at $x = 2$. Sign change: $f''(1) = -6 < 0$ and $f''(3) = 6 > 0$ — confirmed sign change from negative to positive. Therefore $x = 2$ is the only inflection point.

WORKED EXAMPLE

Q6. $\int x \ln x \, dx =$

(A) $\frac{x^2}{2} \ln x - \frac{x^2}{4} + C$ **(B)** $\frac{x^2 \ln x}{2} + C$ **(C)** $x \ln x - x + C$ **(D)** $\frac{x^2}{4}(\ln x - 1) + C$

Answer: (A) IBP with $u = \ln x$, $dv = x\,dx$:

$\int x \ln x \, dx = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} \, dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} + C$

WORKED EXAMPLE

Q7. The general solution of $\frac{dy}{dx} = 2y$ is:

(A) $y = Ce^x$ **(B)** $y = Ce^{2x}$ **(C)** $y = 2Ce^x$ **(D)** $y = x^2 + C$

Answer: (B) Separate: $\frac{dy}{y} = 2\,dx \Rightarrow \ln|y| = 2x + k \Rightarrow y = Ce^{2x}$.

WORKED EXAMPLE

Q8. Which of the following is the correct Maclaurin expansion of $\cos(2x)$ up to the x^4 term?

(A) $1 - 2x^2 + \frac{2x^4}{3}$ **(B)** $1 - 2x^2 + \frac{x^4}{3}$ **(C)** $1 - x^2 + \frac{x^4}{6}$ **(D)** $1 - 4x^2 + \frac{8x^4}{3}$

Answer: (A) Substitute $2x$ into the standard $\cos x$ series:

$\cos(2x) = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} = 1 - \frac{4x^2}{2} + \frac{16x^4}{24} = 1 - 2x^2 + \frac{2x^4}{3}$

Common error: substituting $x \to 2x$ but forgetting to apply the exponents — writing $\frac{2x^2}{2}$ instead of $\frac{(2x)^2}{2} = \frac{4x^2}{2}$.

WORKED EXAMPLE

Q9. If $f''(x) > 0$ on (a, b) then on (a, b), f is:

(A) decreasing **(B)** increasing **(C)** concave down **(D)** concave up

Answer: (D) $f'' > 0$ means the gradient f' is increasing, which is the definition of concave up. The sign of f'' says nothing directly about whether f is increasing or decreasing (that depends on f').

WORKED EXAMPLE

Q10. A particle moves with $v(t) = t^2 - 3t$. The particle first comes to rest at $t =$

(A) 0 **(B)** 1 **(C)** 3 **(D)** 1.5

Answer: (C) $v(t) = 0 \Rightarrow t(t-3) = 0 \Rightarrow t = 0$ or $t = 3$. At $t = 0$ the particle starts from rest, so the **first** time it **comes** to rest again is at $t = 3$.

WORKED EXAMPLE

Q11. The area enclosed by $y = e^x$ and the lines $x = 0, x = 1, y = 0$ is:

(A) e **(B)** $e - 1$ **(C)** $e + 1$ **(D)** 1

Answer: (B) $\int_0^1 e^x\,dx = [e^x]_0^1 = e^1 - e^0 = e - 1.$

WORKED EXAMPLE

Q12. The volume generated by rotating $y = 2x, 0 \le x \le 3$, about the x-axis is:

(A) 12π **(B)** 24π **(C)** 36π **(D)** 72π

Answer: (C) $V = \pi \int_0^3 (2x)^2\,dx = \pi \int_0^3 4x^2\,dx = 4\pi \left[\frac{x^3}{3}\right]_0^3 = 4\pi \cdot 9 = 36\pi.$

WORKED EXAMPLE

Q13. The partial fraction decomposition of $\frac{5}{(x-1)(x+4)}$ is:

(A) $\frac{1}{x-1} - \frac{1}{x+4}$ **(B)** $\frac{1}{x-1} + \frac{1}{x+4}$ **(C)** $\frac{5}{x-1} - \frac{5}{x+4}$ **(D)** $\frac{1}{x-1} + \frac{4}{x+4}$

Answer: (A) Let $\frac{5}{(x-1)(x+4)} = \frac{A}{x-1} + \frac{B}{x+4}$.

$5 = A(x+4) + B(x-1)$. Set $x = 1$: $5 = 5A \Rightarrow A = 1$. Set $x = -4$: $5 = -5B \Rightarrow B = -1$.

WORKED EXAMPLE

Q14. If $\int_0^a x^2\,dx = 9$, then $a =$

(A) 3 **(B)** $\sqrt{3}$ **(C)** $\sqrt[3]{27}$ **(D)** 9

Answer: (A) $\left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3} = 9 \Rightarrow a^3 = 27 \Rightarrow a = 3.$

Note: **(C)** is also equal to 3, since $\sqrt[3]{27} = 3$ — but **(A)** is the standard form.

WORKED EXAMPLE

Q15. Using the Maclaurin series for $\ln(1+x)$, the approximate value of $\ln(1.1)$ to 4 decimal places (using up to the x^4 term) is:

(A) 0.0953 **(B)** 0.1000 **(C)** 0.0900 **(D)** 0.0909

Answer: (A) With $x = 0.1$:

$\ln(1.1) \approx 0.1 - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} = 0.1 - 0.005 + 0.000333 - 0.000025 = 0.095308$

Rounded to 4 d.p.: 0.0953.

WORKED EXAMPLE

Q16. For $f(x) = x^3 - 3x$, the local minimum value is:

(A) -2 **(B)** 0 **(C)** 2 **(D)** 3

Answer: (A) $f'(x) = 3x^2 - 3 = 3(x-1)(x+1) = 0 \Rightarrow x = \pm 1.$

$f''(x) = 6x$. At $x = 1$: $f'' = 6 > 0$ (local min). $f(1) = 1 - 3 = -2$.

Key Formulas

MEMORISE THIS

Complete Calculus Formula Reference

Limits

Formula	Notes
$\lim_{x\to 0} \frac{\sin x}{x} = 1$	Fundamental limit
$\lim_{x\to 0} \frac{e^x - 1}{x} = 1$	Derived from series
L'Hôpital: $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$	Only for $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Differentiation Rules

Rule	Formula
Power	$\frac{d}{dx}[x^n] = nx^{n-1}$
Product	$\frac{d}{dx}[uv] = u'v + uv'$
Quotient	$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{u'v - uv'}{v^2}$
Chain	$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

Implicit: $y^n ny^{n-1}\frac{dy}{dx}$

Key Derivatives

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
e^x	e^x
$\ln x$	$1/x$
$\arcsin x$	$1/\sqrt{1-x^2}$
$\arctan x$	$1/(1+x^2)$

Integration

Rule / Formula	Notes
$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$	$n \neq -1$
$\int \frac{1}{x} \, dx = \ln\|x\| + C$	Absolute value required
$\int e^x \, dx = e^x + C$	
$\int \sin x \, dx = -\cos x + C$	Note sign
$\int \cos x \, dx = \sin x + C$	
$\int \sec^2 x \, dx = \tan x + C$	
$\int u \, dv = uv - \int v \, du$	Integration by parts
Substitution: $\int f(g(x))g'(x) \, dx = \int f(u) \, du u = g(x)$	
$\int \frac{1}{x^2+a^2} \, dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$	
$\int \frac{1}{\sqrt{a^2-x^2}} \, dx = \arcsin\left(\frac{x}{a}\right) + C$	

Applications

Quantity	Formula
Tangent gradient at $x = a$	$m = f'(a)$
Normal gradient at $x = a$	$m = -1/f'(a)$
Area under curve	$A = \int_a^b \|f(x)\| \, dx$
Area between curves	$A = \int_a^b [f(x) - g(x)] \, dx$
Volume of revolution (x-axis)	$V = \pi \int_a^b [f(x)]^2 \, dx$
Displacement from v	$s = \int v \, dt$
Distance from v	$d = \int \|v\| \, dt$

Differential Equations

Type	Method
Separable: $\frac{dy}{dx} = f(x)g(y)$	$\int \frac{dy}{g(y)} = \int f(x)\,dx$
Exponential growth/decay	$N = N_0 e^{\pm kt}$
Newton's cooling: $\frac{d\theta}{dt} = -k\theta$	$\theta = \theta_0 e^{-kt}, \theta = T - T_{\text{env}}$

Maclaurin Series

Function	Series	Interval
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots$	All x
$\sin x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots$	All x
$\cos x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots$	All x
$\ln(1+x)$	$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$	$-1 < x \leq 1$

Mixed Practice — Exam Style

IB TIP

How to use this section: Unlike topic-specific practice, these questions are interleaved — they mix all topics from this guide in random order. Before answering, identify *which concept or topic area* the question is testing. This is exactly the skill you need on Paper 2 and Paper 3, where you don't know in advance which topic each question covers.

1. **[Integration]** Evaluate $\int_0^1 x e^{x^2}\,dx$.

   A. $\frac{e-1}{2}$

   B. $e - 1$

   C. $\frac{1}{2}$

   D. $e^2 - 1$

2. **[Limits]** Evaluate $\lim_{x \to 0} \frac{\sin 3x}{5x}$.

   A. 0

   B. $\frac{3}{5}$

   C. 1

D. $\frac{5}{3}$

3. **[Differential Equations]** The rate of change of a population P satisfies $\frac{dP}{dt} = 0.04P$. If $P(0) = 500$, find $P(10)$.

A. $500 + 0.04(10)$

B. $500e^{0.4}$

C. $500e^{4}$

D. $500(1.04)^{10}$

4. **[Applications of Derivatives]** A function f has $f'(x) = (x-2)^2(x+1)$. Which statement is correct?

A. f has a local minimum at $x = 2$ and a local maximum at $x = -1$

B. f has a local minimum at $x = -1$ only; $x = 2$ is a stationary point of inflection

C. f has local minima at both $x = -1$ and $x = 2$

D. f has a local maximum at $x = -1$ and no stationary point at $x = 2$

5. **[Maclaurin Series]** The first three non-zero terms of the Maclaurin series of $\cos(2x)$ are:

A. $1 - 2x^2 + \frac{2x^4}{3}$

B. $1 - 2x + 2x^2$

C. $2x - \frac{8x^3}{6} + \frac{32x^5}{120}$

D. $1 + 2x^2 - \frac{2x^4}{3}$

6. **[Differentiation Rules]** Find $\frac{dy}{dx}$ if $y = \ln(\sin x)$.

A. $\frac{1}{\sin x}$

B. $\cos x$

C. $\cot x$

D. $-\cot x$

7. **[Volumes of Revolution]** The region bounded by $y = \sqrt{x}$, the x-axis, and $x = 4$ is rotated 2π radians about the x-axis. The exact volume is:

A. 4π

B. 8π

C. 16π

D. 32π

8. **[Limits — L'Hôpital]** Use L'Hôpital's rule to evaluate $\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$.

A. 0

B. $\frac{1}{4}$

C. $\frac{1}{2}$

D. 1

9. **[Differential Equations]** The general solution of $\frac{dy}{dx} = \frac{x}{y}$ is:

A. $y = x + C$

B. $y^2 = x^2 + C$

C. $y = \frac{x^2}{2} + C$

D. $\ln y = \ln x + C$

10. **[Integration by Parts]** Evaluate $\int x \cos x \, dx$.

A. $x \sin x + \cos x + C$

B. $x \sin x - \cos x + C$

C. $-x \sin x + \cos x + C$

D. $\sin x - x \cos x + C$

▶ Show Answers

IB Math IA Ideas — Calculus

IB TIP

Exploration topics from this chapter:

- **Modelling drug concentration** — Use a first-order differential equation $\frac{dC}{dt} = -kC$ to model how a drug is eliminated from the bloodstream. Investigate how the elimination constant k varies across drugs, optimise dosing intervals to keep concentration within a therapeutic window, and compare your model against real pharmacokinetic data.

- **Lorenz curves and the Gini coefficient** — A Lorenz curve $L(x)$ describes income distribution; the Gini coefficient $G = 1 - 2\int_0^1 L(x)\,dx$ measures inequality. Download World Bank income-share data for two countries and use numerical integration to calculate and compare their Gini coefficients. Extend by fitting a functional form to the curve.

- **The Brachistochrone problem** — Find the curve of fastest descent between two points under gravity. Derive the cycloid parametrically and verify it is faster than a straight line by comparing definite integrals of travel time. This connects differentiation, integration, and parametric equations in one elegant problem.

- **Population growth: logistic vs exponential** — The logistic model $\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$ accounts for carrying capacity. Source census data for a real population, solve both ODEs analytically, and use least-squares or residual analysis to determine which model fits better.

- **Optimising packaging** — Minimise the surface area of a cylindrical tin for a fixed volume using $\frac{dS}{dr} = 0$. Extend to elliptical cross-sections, conical lids, or prism shapes, and compare your theoretical optimum against the dimensions of real commercial products.

- **Volumes of revolution in architecture** — Model a dome, arch, or vase using a curve $y = f(x)$ and compute the volume using $V = \pi\int_a^b [f(x)]^2\,dx$. Photograph a real object, digitise its profile, fit a function, and compare the calculated volume to the manufacturer's stated capacity.

- **The Mean Value Theorem and speed cameras** — Prove rigorously that if a car's average speed between two cameras exceeds the limit, then by the Mean Value Theorem it must have instantaneously exceeded the limit at some point. Extend to discuss the mathematics behind average-speed enforcement and how it differs from point-speed checks.

Tip: A strong IA has a clear personal engagement angle. Pick a topic that connects to something you genuinely find interesting — sports, medicine, economics, or architecture — and let the mathematics serve your question, not the other way around.

May 2026 Prediction Questions

⚠ EXAM ALERT

These are NOT official IB questions. These are trend-based practice questions written to reflect the topic areas and question styles most likely to appear on the May 2026 IB Math AA HL Paper 2. Based on recent exam patterns (2022–2025), expect heavy weighting on: integration techniques, differential equations, optimization, and Maclaurin series.

WORKED EXAMPLE

Question 1 [Integration by Parts] [~7 marks]

Evaluate the definite integral

$\int_0^\pi x \sin x \, dx$

showing all working. State the integration technique used.

▶ Show Solution

WORKED EXAMPLE

Question 2 [Optimization] [~8 marks]

A closed cylindrical tin has a fixed volume of $V = 250\pi \text{ cm}^3$. The total surface area of the tin is S.

(a) Show that $S = 2\pi r^2 + \dfrac{500\pi}{r}$, where r is the radius of the base in cm.

(b) Find the value of r that minimizes S, and verify it is a minimum.

(c) Find the minimum surface area, giving your answer in exact form.

▶ Show Solution

WORKED EXAMPLE

Question 3 [Separable Differential Equation] [~7 marks]

Consider the differential equation

$\frac{dy}{dx} = \frac{2x}{y+1}, \quad y > -1$

(a) Find the general solution, expressing y explicitly in terms of x.

(b) Find the particular solution satisfying $y(0) = 2$.

▶ Show Solution

WORKED EXAMPLE

Question 4 [Maclaurin Series] [~7 marks]

(a) Write down the Maclaurin series for $\sin u$ up to and including the term in u^5.

(b) Hence find the first three non-zero terms of the Maclaurin series for $\sin(x^2)$.

(c) State the interval of validity of the series in part (b).

(d) Use the series from part (b) to find an approximation for $\displaystyle\int_0^{0.5} \sin(x^2)\, dx$, giving your answer to 4 significant figures.

▶ Show Solution

WORKED EXAMPLE

Question 5 [Area Between Curves] [~8 marks]

The curves $f(x) = x^2 - 2x$ and $g(x) = 4 - x^2$ intersect at two points.

(a) Find the x-coordinates of the two intersection points.

(b) Find the exact area of the region enclosed between the two curves.

▶ Show Solution